Chapter 4 The physical and flow properties of blood and other fluids

4.1 Physical properties of blood

Blood is a viscous fluid mixture consisting of plasma and cells. Table 4.1 summarizes the most important physical properties of blood. Recall that the chemical composition of the plasma was previously shown in Table 3.2. Proteins represent about 7–8 wt% of the plasma. The major proteins found in plasma are albumin (MW = 69,000; 4.5 g 100 mL\(^{-1}\)), the globulins (MW = 35,000–1,000,000; 2.5 g 100 mL\(^{-1}\)), and fibrinogen (MW = 400,000; 0.3 g 100 mL\(^{-1}\)). Albumin has a major role in regulating the pH and the colloid osmotic pressure of blood. The so-called alpha and beta globulins are involved in solute transport, whereas the gamma globulins are the antibodies that fight infection and form the basis of the humoral component of the immune system. Fibrinogen, through its conversion to long strands of fibrin, has a major role in the process of blood clotting. Serum is simply the fluid remaining after blood is allowed to clot. For the most part, the composition of serum is the same as that of plasma, with the exception that the clotting proteins, primarily fibrinogen, and the cells have been removed.

4.2 Cellular components

The cellular component of blood consists of three main cell types. The most abundant cells are the red blood cells (RBCs) or erythrocytes comprising about 95% of the cellular component of blood. Their major role is the transport of oxygen by the hemoglobin contained within the RBC. Note from Table 4.1 that the density of an RBC is higher than that of plasma. Therefore, in a quiescent fluid, the RBCs will tend to settle. The RBC volume fraction is called the hematocrit and typically varies between 40% and 50%. The true hematocrit (H) is about 96% of the measured hematocrit (Hct).

The RBC has a unique shape described as a biconcave discoid. Figure 4.1 illustrates the size of the RBC and Table 4.2 summarizes typical dimensions. RBCs can form stacked coin-like structures called rouleaux. Rouleaux can also clump together to form larger RBC structures called aggregates. Both rouleaux and aggregates break apart under conditions of increased blood flow or higher shear rates.

Platelets are the next most abundant cell type, comprising about 4.9% of the blood cell volume. The platelets are major players in blood coagulation and hemostasis, which is the prevention of blood loss. The remaining 0.1% of the cellular component of blood consists of the white blood cells (WBCs) or leukocytes, which form the basis of the cellular component of the immune system. Since the WBCs and platelets only comprise about 5% of the cellular component of blood, their effect on the macroscopic flow characteristics of blood is negligible.
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4.3 Rheology

The field of rheology concerns the deformation and flow behavior of fluids. The prefix *rheo*- is from the Greek and refers to something that flows. Due to the particulate nature of blood, we expect the rheological behavior of blood to be somewhat more complex than a simple fluid such as water.

Our understanding of the flow behavior of fluids begins by exploring the relationship between *shear stress* \( \tau \) and the *shear rate* \( \dot{\gamma} \). To develop this relationship, consider the situation shown in Figure 4.2. A fluid is contained between two large parallel plates both of area A. The plates are separated by a small distance equal to \( h \). Initially the system is at rest. At time \( t = 0 \), the lower plate is set into

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole blood</td>
<td></td>
</tr>
<tr>
<td>pH</td>
<td>7.35–7.40</td>
</tr>
<tr>
<td>Viscosity (37°C)</td>
<td>3.0 cP (at high shear rates)</td>
</tr>
<tr>
<td>Specific gravity (25/4°C)</td>
<td>1.056</td>
</tr>
<tr>
<td>Venous hematocrit</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>0.47</td>
</tr>
<tr>
<td>Female</td>
<td>0.42</td>
</tr>
<tr>
<td>Whole blood volume</td>
<td>(~78 \text{ mL kg}^{-1} \text{ body weight})</td>
</tr>
<tr>
<td>Plasma or serum</td>
<td></td>
</tr>
<tr>
<td>Colloid osmotic pressure</td>
<td>(~330 \text{ mm H}_2\text{O})</td>
</tr>
<tr>
<td>pH</td>
<td>7.3–7.5</td>
</tr>
<tr>
<td>Viscosity (37°C)</td>
<td>1.2 cP</td>
</tr>
<tr>
<td>Specific gravity (25/4°C)</td>
<td>1.0239</td>
</tr>
<tr>
<td>Erythrocytes (RBCs)</td>
<td></td>
</tr>
<tr>
<td>Specific gravity (25/4°C)</td>
<td>1.098</td>
</tr>
<tr>
<td>Count</td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>(~5.4 \times 10^9 \text{ mL}^{-1} \text{ whole blood})</td>
</tr>
<tr>
<td>Female</td>
<td>(~4.8 \times 10^9 \text{ mL}^{-1} \text{ whole blood})</td>
</tr>
<tr>
<td>Average life span</td>
<td>120 days</td>
</tr>
<tr>
<td>Production rate</td>
<td>(~4.5 \times 10^7 \text{ mL}^{-1} \text{ whole blood/day})</td>
</tr>
<tr>
<td>Hemoglobin concentration</td>
<td>0.335 g mL(^{-1}) of erythrocyte</td>
</tr>
<tr>
<td>Leukocytes</td>
<td></td>
</tr>
<tr>
<td>Count</td>
<td>(~7.4 \times 10^6 \text{ mL}^{-1} \text{ whole blood})</td>
</tr>
<tr>
<td>Diameter</td>
<td>7–20 (\mu) m</td>
</tr>
<tr>
<td>Platelets</td>
<td></td>
</tr>
<tr>
<td>Count</td>
<td>(~2.8 \times 10^8 \text{ mL}^{-1} \text{ whole blood})</td>
</tr>
<tr>
<td>Diameter</td>
<td>(~2–5 \mu) m</td>
</tr>
</tbody>
</table>

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motion in the x direction at a constant velocity $V$. The velocity of the lower plate is sufficiently low such that the fluid motion is not turbulent, i.e., macroscopic or convective mixing of the fluid in the y direction does not occur. Rather, the fluid motion is laminar. In laminar flow, the fluid flows without any mixing in the y direction. This means that adjacent layers of fluid will slide past one another in a manner analogous to what is seen when a deck of playing cards is deformed.

The fluid velocity is a vector and will have three components, which in these Cartesian coordinates, are $v_x$, $v_y$, and $v_z$. For the situation shown in Figure 4.2, $v_y$ and $v_z$ are both zero; $v_y$ is zero because the plates extend to a great distance in the z direction and the lower plate only moves in the x direction, so we conclude that there is no flow of fluid in the z direction; $v_z$ is zero since there is no motion of the lower or upper plate in the y direction and there are no holes in either of the plates that would cause the fluid to flow in the y direction. So we see that for the situation shown in Figure 4.2, the only velocity component is that in the x direction, i.e., $v_x$, and this will depend, in general, on $(x,y,z,t)$.

Since the lower plate is set into motion at $t = 0$, and the fluid initially is not moving, there will be a transient period during which successive layers of the fluid will be set into motion. This means that

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Table 4.2 Dimensions of the Normal Red Blood Cell

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter</td>
<td>$8.1 \pm 0.43 \mu m$</td>
</tr>
<tr>
<td>Greatest thickness</td>
<td>$2.7 \pm 0.15 \mu m$</td>
</tr>
<tr>
<td>Least thickness</td>
<td>$1.0 \pm 0.3 \mu m$</td>
</tr>
<tr>
<td>Surface area</td>
<td>$138 \pm 17 \mu m^2$</td>
</tr>
<tr>
<td>Volume</td>
<td>$95 \pm 17 \mu m^3$</td>
</tr>
</tbody>
</table>

the flow is inherently unsteady and dependent on time, i.e., t. The plate separation, i.e., h, is constant in the x direction so this means that \( v_x \) will not depend on x. Since the lower plate is moving at V and the upper plate is stationary, \( v_x \) will depend only on y. However, since the plates are very large in the z direction compared to their separation, i.e., h, \( v_x \) at a given y and t will be the same for any value of z. So we conclude that \( v_x \) can only be a function of y and t, i.e., \( v_x(y, t) \).

As time proceeds, momentum is transferred in the y direction to successive layers of fluid from the lower plate that is in motion in the x direction. In laminar flow, momentum transport occurs by diffusion and we say that momentum “flows or diffuses” from a region of high velocity to a region of low velocity. After a sufficient length of time, a steady-state velocity profile is obtained, i.e., a linear function of y, i.e., \( v_x(y, t) = \frac{V}{h}(h - y) \).

At steady state, a constant force (F) must be applied to overcome the resistance of the fluid and maintain the motion of the lower plate at velocity V. For the situation shown in Figure 4.2, the shear stress on the lower plate is defined as \( F/A \) and is given the symbol \( \tau_{yx} \), where the subscript \( yx \) denotes the viscous flux* of x momentum in the y direction (Bird et al., 2002). The shear stress, i.e., \( \tau_{yx} \), can also be interpreted as a force acting in the x direction on a surface that is perpendicular to the y direction. The shear rate at any position y in the fluid is defined as \( -\frac{dv_x(y)}{dy} = \frac{V}{h} = \dot{\gamma} \). The shear rate is given the symbol \( \dot{\gamma} \). Notice that shear rate has units of reciprocal time. The shear rate is also the same as the strain rate or the rate of deformation.

Newton’s law of viscosity states that for laminar flow (nonturbulent) the shear stress is proportional to the shear rate. The proportionality constant is called the viscosity, \( \mu \), which is a physical property of the fluid and is a measure of the flow resistance of the fluid. Viscosity is usually expressed in the following units where 1 P (poise) = 100 cP (centipoise) = 1 g cm\(^{-1}\) s\(^{-1}\) = 1 dyne cm s\(^{-2}\) = 0.1 N s m\(^{-2}\) = 0.1 Pa s. Also 1 cP = 0.001 Pa s. For the situation shown in Figure 4.2, Newton’s law of viscosity can be stated as follows:

\[
\frac{F}{A} = \tau_{yx} = \frac{V}{h} = \mu \dot{\gamma}
\]

Most simple homogeneous liquids and gases obey this law and are called Newtonian fluids.

* A flux is a flow that has been normalized with respect to the area that is perpendicular to the direction of the flow.
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For more complicated geometries, the steady-state velocity profile is not linear. However, Newton’s law of viscosity can be stated at any point in the laminar flow field. In Cartesian coordinates (x, y, z), and in cylindrical coordinates (r, \(\theta\), z), we can write Newton’s law of viscosity as

\[
\tau_{yx} = -\mu \frac{dv_x}{dy} = \mu \gamma
\]

(4.2a)

\[
\tau_{rz} = -\mu \frac{dv_z}{dr} = \mu \gamma
\]

(4.2b)

where

\(v_x\) is the velocity in the x direction at position y
\(v_z\) is the velocity in the z direction at position r

These equations tell us that the momentum flows in the direction of decreasing velocity. The velocity gradient is therefore the driving force for momentum transport much like the temperature gradient is the driving force for heat transfer, and the concentration gradient is the driving force for mass transport.

A fluid whose shear stress–shear rate relationship does not follow Equation 4.2a and b is known as a non-Newtonian fluid. Figure 4.3 illustrates the type of shear stress–shear rate relationships that are typically observed for non-Newtonian fluids. The Newtonian fluid is shown for comparison. Note from Equation 4.2a and b that the shear stress–shear rate relationship for a Newtonian fluid is linear with the slope equal to the viscosity. For non-Newtonian fluids, we can also define the observed or apparent viscosity for given values of \(\tau_{yx}\) and \(\gamma\). Hence, from Equation 4.2a and b, and for given values of \(\tau_{yx}\) and \(\gamma\), the apparent viscosity \(\mu_{\text{apparent}}\) for a non-Newtonian fluid is given by \(\tau_{yx}/\gamma\) or \(\tau_{rz}/\gamma\). Note that the apparent viscosity is not constant and will depend on the shear rate, \(\gamma\).

A dilatant fluid thickens or has an increase in apparent viscosity as the shear rate increases. An example of a dilatant fluid is a solution of cornstarch and water, which allows one to “walk on water.” As a person quickly walks or runs across this fluid, each step induces a high shear rate in the fluid surrounding the foot increasing the fluid viscosity and the resulting shear stress that is generated.

![Figure 4.3 Types of shear stress–shear rate relationship.](image_url)
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supports the person and they do not sink into the fluid. A pseudoplastic fluid, on the other hand, tends to thin out, or its apparent viscosity decreases, with an increase in shear rate. Paint is an example of a pseudoplastic fluid, since it thins out as it is brushed or quickly sheared as it is applied to a surface.

Heterogeneous fluids that contain a particulate phase that forms aggregates at low rates of shear exhibit a yield stress, \( \tau_y \). The yield stress must be exceeded in order to get the material to flow. Two types of fluids that exhibit this behavior are the Bingham plastic and the Casson fluid. The Casson fluid has been used to describe liquids that contain particulates that also aggregate, forming large complex structures. An example of this type of fluid is the ink used in ball point pens. When the pen is not being used, the ink thickens and cannot flow out of the pen. As one writes, the rotating ball at the point of the pen shears the fluid, thinning it out so it can be applied to the paper. In the case of the Bingham plastic, once the yield stress is exceeded, the fluid behaves as if it were Newtonian. For the Casson fluid, we see that as the shear rate increases, the apparent viscosity decreases, indicating that the particulate aggregates are getting smaller and smaller and at some point the fluid behaves as a Newtonian fluid. Blood is a heterogeneous fluid, with the particulates consisting primarily of RBCs. As mentioned earlier and shown in Figure 4.1, the RBCs form rouleaux and aggregates at low shear rates. We will see in the ensuing discussion that the rheology of blood behaves like that of the Casson fluid.

4.4 The capillary viscometer and laminar flow in tubes

The simplest approach for examining the shear stress-shear rate behavior of blood or other fluids is through the use of the capillary viscometer shown in Figure 4.4. The diameter of the capillary tube is typically on the order of 500 \( \mu \)m. To eliminate entrance and exit effects, the ratio of the capillary length (L) to its radius (R) should be greater than 100 (Rosen, 1993). For a given flow rate of fluid, \( Q \), the pressure drop across the viscometer, \( \Delta P = P_0 - P_L \), is measured. From this information, it is possible to deduce an analytical expression for the shear stress-shear rate relationship, i.e., \( \dot{\gamma} = \dot{\gamma} (\tau_y) \).

Analysis of the flow of a fluid in the horizontal cylindrical capillary tube illustrated in Figure 4.4 requires the use of cylindrical coordinates (r, \( \theta \), z). In cylindrical coordinates, the components of the

![Figure 4.4 Apparent viscosity of blood at 37°C.](image)

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velocity vector are $v_r$, $v_\theta$, and $v_z$. With steady-state flow, i.e., no dependence on time, we can make the following simplifying assumptions: the length of the tube ($L$) is much greater than the tube radius ($R$) (i.e., $L/R \sim 100$) to eliminate entrance effects; the flow is incompressible (i.e., constant density) and isothermal (i.e., constant temperature); no external forces acting on the fluid; no holes in the tube so that there is no radial velocity component $v_r$; no axisymmetric flow or no swirls so that the tangential velocity $v_\theta$ is also zero; and the no-slip condition at the wall requires that $v_z = 0$ at $r = R$. Continuity or conservation of mass for an incompressible fluid with these assumptions therefore requires that only an axial velocity component exists and it will be a function of the tube radius only, therefore $v_z = v_z(r)$.

Now consider in Figure 4.4 a cylindrical volume of fluid of radius $r$ and length $\Delta z$. For steady flow, the viscous force acting to retard the fluid motion, i.e., $\tau_r 2\pi r \Delta z$, must be balanced by the force developed by the pressure drop acting on the volume of fluid of length $\Delta z$, i.e., $\pi r^2 (P_0 - P_L) / L$. Equating these forces, dividing by $\Delta z$, and taking the limit as $\Delta z \to 0$, the following equation is obtained for the shear stress distribution for the fluid flowing within the capillary tube:

$$\frac{\tau_r (r)}{r} = -\frac{1}{2} \frac{dP}{dz} = \frac{P_0 - P_L}{2L}$$

(4.3)

In Equation 4.3, the first term on the left-hand side of the equation is only a function of $r$ and middle term of the equation is only a function of $z$. Hence, both of these terms must equal the same constant, and this means that $-dP/dz$ is equal to the pressure drop per unit length of the tube, i.e., $(P_0 - P_L)/L$.

We note that the shear stress vanishes at the centerline of the capillary and achieves its maximum value, $\tau_w$, at the wall where $r = R$. The wall shear stress is easily calculated by the following equation in terms of the measured pressure drop $(P_0 - P_L)$ and the tube dimensions (R and L):

$$\tau_w = \tau_{rz}(R) = -\frac{R}{2} \frac{dP}{dz} = \frac{(P_0 - P_L)R}{2L}$$

(4.4)

It should also be pointed out that Equations 4.3 and 4.4 hold whether the fluid is Newtonian or non-Newtonian. Nothing has been said so far about the relationship between a particular fluid’s shear stress and shear rate. Using Equation 4.4, one can rewrite Equation 4.3 to give the shear stress in terms of the wall shear stress and the fractional distance from the centerline of the capillary tube, i.e., $r/R$:

$$\tau_{rz}(r) = \frac{\tau_w r}{R}$$

(4.5)

4.4.1 Hagen–Poiseuille equation for laminar flow of a Newtonian liquid in a cylindrical tube

Let’s first consider the flow of a Newtonian liquid in a cylindrical tube. Liquids are considered to be incompressible and their density is constant. For a Newtonian liquid flowing in a cylindrical horizontal tube, we know from Equation 4.2b that $\tau_{rz} = -\mu (dv_z/dr)$. Substituting this relation for $\tau_{rz}$ into Equation 4.3, we obtain

$$\tau_{rz} = -\mu \frac{dv_z}{dr} = \frac{(P_0 - P_L) r}{2L}$$

(4.6)
This equation can be easily integrated using the no-slip boundary condition that says that, at \( r = R \), \( v_z = 0 \). The result of this integration provides an equation that describes the velocity profile for laminar flow of a Newtonian liquid in a cylindrical tube. The following equation shows that the velocity profile will have a parabolic shape:

\[
v_z(r) = \frac{(P_0 - P_L)R^2}{4 \mu L} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]
\]  

(4.7)

Note that the maximum velocity \( (V_{\text{maximum}}) \) occurs at the centerline of the cylindrical tube, where \( r = 0 \); hence the maximum velocity is given by

\[
V_{\text{maximum}} = \frac{R^2 (P_0 - P_L)}{4 \mu L}
\]  

(4.8)

The total volumetric flow rate of the liquid in a cylindrical tube is given by

\[
Q = 2\pi \int_0^R v_z(r) r dr
\]  

(4.9)

Substituting Equation 4.7 for \( v_z(r) \) into Equation 4.9 and performing the integration gives

\[
Q = \frac{\pi R^4 (P_0 - P_L)}{8 \mu L} = \frac{\pi d^4 (P_0 - P_L)}{128 \mu L}
\]  

(4.10)

where \( d \) is the tube diameter.

Equation 4.10 is the Hagen–Poiseuille law for laminar flow of a Newtonian liquid in a cylindrical tube. It provides a simple relationship between the volumetric flow rate in the tube given by \( Q \), and the pressure drop between the entrance and the exit of the tube, i.e., \( P_0 - P_L \), for the given dimensions of the tube \( R \) and \( L \), and the fluid viscosity \( \mu \).

Using the analogy that flow is proportional to a driving force divided by a resistance, we see that the driving force is the pressure drop, \( P_0 - P_L \), and the resistance is given by \( 8\mu L/\pi R^4 \). This resistance term is very important since it shows that for laminar flow within blood vessels, small changes in the radius of the vessel can have a significant effect on the blood flow rate for a given pressure drop. To achieve this control of the blood flow rate, the arterioles, the smallest elements of the arterial system with diameters less than 100 \( \mu \text{m} \), consist of an inner endothelial cell lining that is surrounded by a layer of vascular smooth muscle cells. Contraction or relaxation of the smooth muscle layer provides a reactive method for controlling arteriole diameter and hence the blood flow rate within organs and tissues.

For laminar flow in a network of tubes or small capillaries, the overall pressure drop across the network is \( P_0 - P_L \), and the resistance for the \( i \)th element of the network is \( 8\mu L_i/\pi R_i^4 \), where \( L_i \) and \( R_i \) are, respectively, the length and radius of the \( i \)th element. We can then draw on the analogy to an electrical circuit with resistors in series and in parallel to solve for the overall flow \( (Q) \) in the network and for the flow \( (Q_i) \) in each element of the network.
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If the entrance of the tube is at a height $Z_0$ above a reference elevation, and the exit of the tube is at a height $Z_L$ relative to the same reference elevation, then the pressure difference term in Equation 4.10 must account for the change in the hydrostatic pressure between the entrance and exit of the tube. Hence, in this case, we can write the pressure difference term in Equation 4.10 as 

$$[(P_0 - P_L) + \rho g (Z_0 - Z_L)]$$

The average velocity ($V_{\text{average}}$) of the fluid flowing within a cylindrical tube is defined as the ratio of the volumetric flow rate ($Q$) to the cross-sectional area (i.e., $A = \pi R^2$) of the tube normal to the flow direction, i.e., $V_{\text{average}} = Q/A$. Using Equation 4.10 for $Q$, the average velocity is then given by

$$V_{\text{average}} = \frac{R^2 (P_0 - P_L)}{8 \mu L} = \frac{1}{2} V_{\text{maximum}}$$

(4.11)

Example 4.1

Equation 4.10 was derived under the assumption of the laminar flow of a Newtonian liquid. What happens if the fluid is a Newtonian gas? Develop an expression for the pressure change over the length of the tube, i.e., $(P_0 - P_L)$, for a gas in laminar flow.

Solution

Unlike a liquid whose density is constant, the density of a gas will change along the flow path because of the decrease in pressure. However, conservation of mass will require that the mass flow rate of the gas in the tube is constant, which means that at any axial location within the tube, we must have that $\dot{m} = \rho \dot{Q}$. Now assuming we have an ideal gas, the gas density is then given by $\rho = P/RT$. If we also assume that the gas flow in the tube is isothermal, then between the tube entrance, i.e., $z = 0$, and any location in the tube, $z$, we have that $\rho(z)/P(z) = \rho_0/P_0$. Now at some location $z$ within the tube, the local volumetric flow rate for a Newtonian gas is given by Equation 4.10, where we have replaced $(P_0 - P_L)/L$ with $-dP/dz$, to account for the change in the gas density as the pressure changes

$$Q(z) = \frac{\dot{m}}{\rho(z)} = \frac{\dot{m} P_0}{\rho_0 P(z)} = \frac{\pi R^4}{8 \mu} \frac{dP}{dz}$$

This is a differential equation that can then be solved for the isothermal pressure change of a gas flowing in a cylindrical tube. Hence,

$$Q_0 = \frac{\dot{m}}{\rho_0} = \left( \frac{\pi R^4}{8 \mu L} \right) \left( \frac{P_0 + P_L}{2} \right) \left( \frac{P_L - P_0}{P_0} \right)$$

(A)

where $Q_0$ is the volumetric flow rate of the gas at the tube entrance.

Gas viscosities at normal pressures are considerably less than that of liquids. For example, the viscosity of water is 1 cP at room temperature, whereas the viscosity of air is only 0.018 cP. Comparing Equation A in Example 4.1 for a gas, with Equation 4.10 for a liquid, with everything but the viscosity the same, we see that the flow rate of the gas in a tube would be about 50 times larger than that of the liquid. Conversely, for the same flow rate of the gas and liquid, the pressure drop for the gas would be about 50 times less than that for the liquid. Hence, in most biomedical engineering applications, the pressure drop for gas flow is usually negligible.
Example 4.2

A hollow fiber module is being designed for a bioreactor application. Mammalian liver cells will be grown in the shell space surrounding the hollow fibers and nutrient media will flow through the inside of the hollow fibers. The module will contain a total of 7000 hollow fibers with inside diameter of 500 μm and length 35 cm. If the pressure drop over the length of these fibers is 25 mmHg, estimate the total flow rate in mL min⁻¹ of the nutrient media through the hollow fiber module. The nutrient media has a viscosity of 0.85 cP and a density of 1 g cm⁻³ and the flow is laminar.

Solution

Since the pressure drop over the length of each hollow fiber has to be the same this means that we can calculate the flow rate for one hollow fiber and then multiply that value by the number of fibers to obtain the total flow rate. Using Equation 4.10, we have

\[
Q_{\text{fiber}} = \frac{\pi(0.05 \text{ cm})^4 \times 25 \text{ mmHg} \times 1 \text{ atm} / 760 \text{ mmHg} \times 101,325 \text{ Pa/atm}}{128 \times 0.00085 \text{ Pas} \times 35 \text{ cm}} = 0.0172 \text{ cm}^3 / \text{s}
\]

\[
Q_{\text{total}} = Q_{\text{fiber}} \times \# \text{ of fibers} = 0.0172 \text{ cm}^3 / \text{s} \times 7000 \times 60 \text{ s/min} = 7218 \text{ mL/min}
\]

4.4.1.1 Laminar flow of a Newtonian fluid through a tube of very short length

If the length of the tube is comparable to the radius of the tube then the Hagen–Poiseuille equation needs to take into account entrance effects. This can happen, e.g., in nanoengineered ultrathin membranes. This means that the velocity profile has to develop in the tube from its flat entrance profile to the parabolic velocity profile given by Equation 4.7. Dagan et al. (1982) obtained a solution to this problem and in this case the volumetric flow rate is given by

\[
Q = \frac{R^4 (P_0 - P_L)}{\mu} \left[3 + \frac{8}{\pi} \left( \frac{L}{R} \right) \right]
\]

If \[\frac{8}{\pi} \left( \frac{L}{R} \right) \gg 3\], then Equation 4.12 reduces to Equation 4.10.

4.4.2 Hagen–Poiseuille equation for laminar flow of a Newtonian liquid in tubes of noncircular cross section

Solutions for laminar flow of a Newtonian fluid through tubes of noncircular cross sections have also been described in the literature (Lamb, 1932; Bird et al., 2002; Lekner, 2007). For laminar flow in the annular space formed by two concentric cylindrical tubes:

\[
Q = \frac{\pi R^4 (P_0 - P_L)}{8 \mu L} \left[ \left(1 - \kappa^4 \right) - \frac{(1 - \kappa^2)^2}{\ln(1/\kappa)} \right]
\]

where

- \(R\) is the radius of the outer tube
- \(\kappa R\) is the radius of the inner tube
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For an ellipsoidal tube cross section of major axis $a$ and minor axis $b$, we have that

$$ Q = \frac{\pi (P_0 - P_L) (ab)^3}{4 \mu L (a^2 + b^2)} $$

(4.14)

For laminar flow through the rectangular slit of height $2H$ formed between two large parallel and horizontal plates of width $W$ and length $L$ in the flow direction

$$ Q = \frac{2}{3} WH \left(\frac{P_0 - P_L}{\mu L}\right) $$

(4.15)

4.5 The Rabinowitsch equation for the flow of a non-Newtonian fluid in a cylindrical tube

Through the use of the data obtained from the capillary viscometer of given length, $L$, and radius, $R$, a general relationship between the shear rate and some function of the shear stress can be determined in terms of the measurable quantities $Q$ and $(P_0 - P_L)$.

Recall that for the capillary viscometer shown in Figure 4.4, we can write the total flow rate $Q$ in terms of the axial velocity profile as follows:

$$ Q = 2\pi \int_0^R v_z(r) r dr $$

(4.16)

Next, we integrate Equation 4.16 by parts to obtain the following equation:

$$ Q = -\pi \int_0^R \left( \frac{dv_z(r)}{dr} \right) r dr $$

(4.17)

Since $\dot{\gamma} = -\frac{dv_z}{dr}$ and from Equation 4.5 we have that $r = R(tau/\tau_w)$, we can show that the following equation is obtained. This is called the Rabinowitsch equation:

$$ Q = \frac{\pi R^3}{\tau_w} \int_0^\tau \dot{\gamma}(\tau_{ax}) \tau_{ax}^2 d\tau_{ax} $$

(4.18)

For data obtained from a given capillary viscometer, the experiments will provide $Q$ as a function of the observed pressure drop, i.e., $P_0 - P_L$. The wall shear stress, $\tau_w$, is related to $P_0 - P_L$ by Equation 4.4. The appropriate shear rate and shear stress relationship, i.e., $\dot{\gamma}(\tau_{ax})$, is the one that best fits the data according to Equation 4.18.

For example, consider the simplest case of a Newtonian fluid. Equation 4.19 provides the relationship for $\dot{\gamma}(\tau_{ax})$:

$$ \dot{\gamma}(\tau_{ax}) = \frac{\tau_{ax}}{\mu} $$

(4.19)
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Substituting this equation into Equation 4.18 and integrating, we can readily obtain the result obtained earlier, i.e., Equation 4.10, for the laminar volumetric flow rate of a Newtonian liquid flowing in a cylindrical tube.

### 4.6 Other useful flow relationships

Some other useful relationships for Newtonian flow in a cylindrical tube may be obtained by combining Equations 4.4 and 4.10. First, we obtain the following expression that relates the wall shear stress to the volumetric flow rate:

\[
\tau_w = \left( \frac{4 \mu}{\pi} \right) \frac{Q}{R^3} \quad (4.20)
\]

Also, by writing Equation 4.19 at the tube wall and using Equation 4.20, it is found that the shear rate at the wall is given by

\[
\dot{\gamma}_w = \frac{4Q}{\pi R^3} = \frac{4V_{\text{average}}}{R} = \frac{8V_{\text{average}}}{d} \quad (4.21)
\]

The reduced average velocity \( \bar{U} \) is related to the wall shear rate and is defined as the ratio of the average velocity and the tube diameter. It is given by the following equation:

\[
\bar{U} = \frac{V_{\text{average}}}{d} = \frac{4Q}{\pi d^3} = \frac{1}{8} \dot{\gamma}_w \quad (4.22)
\]

The Rabinowitsch equation (4.18) can also be rewritten in terms of \( \bar{U} \), i.e.,

\[
\bar{U} = \frac{4Q}{\pi d^3} = \frac{1}{2\tau_w} \int_0^{\tau_w} \dot{\gamma}_w (\tau_{xz}) \tau_{xz}^2 d\tau_{xz} \quad (4.23)
\]

This equation predicts that the reduced average velocity is only a function of the wall shear stress.

Table 4.3 provides a summary of these key flow equations.

**Example 4.3**

A large artery can have a diameter of about 0.5 cm, whereas a large vein may have a diameter of about 0.8 cm. The average blood velocity in these arteries and veins is, respectively, on the order of 40 and 20 cm s\(^{-1}\). Calculate for these blood flows the wall shear rate, \( \dot{\gamma}_w \).

**Solution**

We can use Equation 4.21 to calculate the wall shear rates.
The physical and flow properties of blood and other fluids

Table 4.3 Summary of Key Flow Equations

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Non-Newtonian</th>
<th>Newtonian</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear stress, $\tau_v =</td>
<td>$\frac{(P_0 - P_L) r}{2L} = \tau_v \frac{r}{R}$</td>
<td>$\frac{(P_0 - P_L) R}{2L} = \tau_v \frac{r}{R}$</td>
</tr>
<tr>
<td>Wall shear stress, $\tau_w =$</td>
<td>$\frac{(P_0 - P_L) R}{2L}$</td>
<td>$\frac{(P_0 - P_L) R}{2L}$</td>
</tr>
<tr>
<td>Shear rate, $\dot{\gamma} =$</td>
<td>$- \frac{dv_z}{dr} = \dot{\gamma} (\tau_v)$</td>
<td>$- \frac{dv_z}{dr} = \frac{\tau_v}{\mu}$</td>
</tr>
<tr>
<td>Volumetric flow rate, $Q =$</td>
<td>$\frac{\pi R^3}{8} \int_{v_{in}}^{v_{out}} \dot{\gamma} (\tau_v) \cdot v_z^2 dv_z$</td>
<td>$\frac{\pi R^4 \Delta P}{8 \mu L}$</td>
</tr>
<tr>
<td>Wall shear stress, $\tau_w =$</td>
<td></td>
<td>$4 \mu Q$</td>
</tr>
<tr>
<td>Wall shear rate, $[\dot{\gamma}_w]$</td>
<td></td>
<td>$4Q$</td>
</tr>
<tr>
<td>Reduced average velocity, $\bar{U}$</td>
<td>$\frac{4Q}{\pi D^2}$</td>
<td>$\frac{4Q}{\pi D^2}$</td>
</tr>
</tbody>
</table>

For the large artery

$$\dot{\gamma}_w = \frac{4 \times 40 \text{ cm}^{-1}}{0.25 \text{ cm}} = 640 \text{ s}^{-1}$$

and for the large vein

$$\dot{\gamma}_w = \frac{4 \times 20 \text{ cm}^{-1}}{0.40 \text{ cm}} = 200 \text{ s}^{-1}$$

**Example 4.4**

Nutrient media is flowing at the rate of 1.5 L min$^{-1}$ in a tube that is 5 mm in diameter. The walls of the tube are covered with antibody-producing cells, and these cells are anchored to the tube wall by their interaction with a special coating material that was applied to the surface of the tube. If one of these cells has a surface area of 500 $\mu$m$^2$, what is the amount of the force that each cell must resist as a result of the flow of this fluid in the tube? Assume the viscosity of the nutrient media is 1.2 cP = 0.0012 Pa s.

**Solution**

We can use Equation 4.20 to find the shear stress acting on the cells that cover the surface of the tube:

$$\tau_w = \left( \frac{4 \times 0.0012 \text{ Pas}}{\pi} \right) \frac{1500 \text{ cm}^2}{\text{min}(0.25 \text{ cm})} \times \frac{1 \text{ min}}{60 \text{ s}} = 2.44 \text{ Pa} = 2.44 \text{ Nm}^{-2}$$
Multiplying \( \tau_w \) by the surface area of a cell gives the force acting on that cell as a result of the fluid flow. Therefore

\[
F_{\text{cell}} = 500 \, \mu m^2 \times \left( \frac{10^{-6} m}{\mu m^2} \right)^2 \times 2.44 \frac{N}{m^2} \times 10^{12} \text{pN} = 1220 \text{pN}
\]

### 4.7 The rheology of blood and the Casson equation

The apparent viscosity of blood as a function of shear rate is illustrated in Figure 4.5 at a temperature of 37\(^\circ\)C. At low shear rates, the apparent viscosity of blood is quite high due to the presence of rouleaux and aggregates. However, at shear rates above about 100–200 s\(^{-1}\), blood behaves as if it were a Newtonian fluid. We then approach the asymptotic high shear rate limit for the apparent viscosity of blood, which is about 3–4 cP.

At high shear rates, where blood is basically a Newtonian fluid, the following equations can be used to express the dependence of the blood viscosity on temperature and hematocrit (Charm and Kurland, 1974):

\[
\mu = \mu_{\text{plasma}} \frac{1}{1 - \alpha H}, \quad \text{for } H \leq 0.6
\]

where

\[
\alpha = 0.070 \exp \left[ 2.49H + \frac{1107}{T} \exp (-1.69H) \right]
\]

In the previous equations, temperature (T) is in K. These equations may be used to a hematocrit of 0.60 and the stated accuracy is within 10%.

#### 4.7.1 The Casson equation

The shear stress-shear rate relationship for blood can be described by the following empirical equation known as the Casson equation:

\[
\tau^{1/2} = \tau_{y}^{1/2} + s \dot{\gamma}^{1/2}
\]

In this equation, \( \tau_y \) is the yield stress and \( s \) is a constant, both of which can be determined from viscometer data. The yield stress represents the fact that a minimum force must be applied to stagnant
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Blood before it will flow. This was illustrated in Figure 4.3. The yield stress for blood at 37°C is about 0.04 dynes cm⁻². It is important to point out, however, that the effect of the yield stress on the flow of blood is small, as the following example will show.

Example 4.5

Estimate the pressure drop in a small blood vessel that is needed to just overcome the yield stress.

Solution

We can use Equation 4.4 to solve for the pressure drop needed to overcome the yield stress ($\tau_y = \tau_w$):

$$(P_0 - P_L)_{\text{min}} = \frac{2L \tau_y}{R}$$

Using the yield stress of 0.04 dynes cm⁻² and an L/R of 200 for a blood vessel, the pressure drop required to just initiate the flow is

$$(P_0 - P_L)_{\text{min}} = 2 \times 200 \times 0.04 \text{ dynes cm}^{-2} \times \frac{1 \text{ bar}}{10^6 \text{ dynes cm}^{-2}} \times 750.061 \text{ mmHg} = 0.012 \text{ mmHg}$$

This result is considerably less than the mean blood pressure, which is on the order of 100 mmHg.

At large values of the shear rate, the apparent viscosity of blood approaches its asymptotic value as shown in Figure 4.5. From Equation 4.25 at large shear rates, we see that the parameter $s$ can be interpreted as the square root of the asymptotic Newtonian viscosity. The asymptotic viscosity of blood is about 3 cP, therefore, the parameter $s$ is $\sqrt{3 \text{cP}} = 1.732(\text{cP})^{1/2}$ or 0.173 (dynes s cm⁻²)¹/₂. It should be stressed that the values of the yield stress and the parameter $s$ will also depend on plasma protein concentrations, hematocrit, and temperature. Therefore, one must be careful to properly calibrate the Casson equation to the actual blood and the conditions used in the viscometer or the situation under study.

4.7.2 Using the Casson equation

For a Casson fluid like blood, we can rearrange Equation 4.25 and solve for the shear rate, i.e., $\dot{\gamma}(\tau_x)$, in terms of the shear stress, $\tau_x$:

$$\dot{\gamma}(\tau_x) = \frac{\sqrt{\tau_x^2 - \tau_y^{1/2}} - \tau_y}{s^2}$$

This equation can be substituted into Equation 4.23 and the resulting equation integrated to obtain the following fundamental equation that describes the flow of the Casson fluid in a cylindrical tube. This equation depends on only two parameters, $\tau_y$ and $s$, which can be determined from experimental data:

$$U = \frac{1}{2s^2} \left[ \tau_w - \frac{4}{7} \sqrt{\tau_w} \sqrt{\tau_y} - \frac{1}{84} \frac{\tau_y^3}{\tau_w^3} + \frac{\tau_y}{3} \right]$$

(4.27)
Using Equation 4.22, we can replace the reduced average velocity in Equation 4.27 with the volumetric flow rate, i.e., $Q$, and obtain

$$Q = \frac{\pi R^4 (P_0 - P_L)}{8 \mu_\infty L} \left[ 1 - \frac{16}{7} \sqrt{\frac{\tau_y}{\tau_w} + \frac{4}{3} \frac{\tau_y}{\tau_w} - \frac{1}{21} \left( \frac{\tau_y}{\tau_w} \right)^4} \right]$$

(4.28)

where

- $\mu_\infty$ is equal to $s^2$ and is the viscosity of the fluid at high shear rates
- $\tau_y$ is given by Equation 4.4

Equation 4.28 is the Casson fluid equivalent to the Hagen–Poiseuille equation, i.e., Equation 4.10.

Example 4.6

Merrill et al. (1965) in a series of classic experiments studied the flow of blood in capillary tubes of various diameters. The blood had a hematocrit of 39.3 and the temperature was 20°C. They measured the pressure drop as a function of the flow rate for five tube diameters ranging from 288 to 850 $\mu$m. When they expressed the measured pressure drops in terms of the wall shear stress (Equation 4.4), and the volumetric flow rates in terms of the reduced average velocity (Equation 4.22), all of the data for the various tube sizes formed, within the experimental accuracy, a single line as predicted by Equation 4.23. From their results they provide the following values of the Casson parameters at 20°C: $\tau_y = 0.0289$ dynes cm$^{-2}$ and $s = 0.229$ (dynes s cm$^{-2}$)$^{1/2}$. Using these values for $\tau_y$ and $s$, show that Equation 4.26 provides an excellent fit to their data summarized in the following table.

<table>
<thead>
<tr>
<th>$\tau_w$, dynes cm$^{-2}$</th>
<th>$\bar{u}$, s$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>0.01</td>
</tr>
<tr>
<td>0.10</td>
<td>0.05</td>
</tr>
<tr>
<td>0.15</td>
<td>0.10</td>
</tr>
<tr>
<td>0.40</td>
<td>0.50</td>
</tr>
<tr>
<td>0.70</td>
<td>1.0</td>
</tr>
<tr>
<td>2.7</td>
<td>5.0</td>
</tr>
<tr>
<td>4.4</td>
<td>10.0</td>
</tr>
<tr>
<td>17.0</td>
<td>50.0</td>
</tr>
<tr>
<td>30.0</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Solution

Figure 4.6 shows a good comparison between the data given in the previous table and the predicted reduced average velocity from Equation 4.27 using the parameter values from Merrill et al. (1965).

4.7.3 The velocity profile for tube flow of a Casson fluid

We can also use the Casson equation to obtain an expression for the axial velocity profile for the flow of blood in a cylindrical tube or vessel. The maximum shear stress, $\tau_y$, is at the wall. If the yield stress, $\tau_y$, is greater than $\tau_w$, then there will be no flow of the fluid. On the other hand, if $\tau_w$ is
greater than $\tau_y$, there will be flow; however, there will be a critical radius ($r_{\text{critical}}$) at which the local shear stress will equal $\tau_y$. From the tube centerline to this critical radius, this core fluid will have a flat velocity profile, i.e., $v_z(r) = v_{\text{core}}$, and will move as if it were a solid body or in what is known as “plug flow.” For the region from the critical radius to the tube wall ($r_{\text{critical}} \leq r \leq R$), Equation 4.3 describes the shear stress distribution as a function of radial position. We can set this equation equal to the shear stress relation provided by the Casson equation (4.25) and rearrange to obtain the following equation for the shear rate:

$$
\dot{\gamma} = -\frac{d v_z(r)}{dr} = \left[ \frac{P_0 - P_L}{2L} r - 2 \tau_y^\frac{1}{2} \left[ \frac{P_0 - P_L}{2L} r \right]^{\frac{1}{2}} + \tau_y \right] \frac{1}{s} \quad (4.29)
$$

This equation can be integrated to find $v_z(r)$ using the boundary condition that at $r = R$, $v_z(R) = 0$, thereby obtaining the following result:

$$
v_z(r) = \frac{R \tau_w}{2s} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] - \frac{8}{3} \sqrt{\frac{\tau_y}{\tau_w}} \left[ 1 - \left( \frac{r}{R} \right) \right]^{\frac{3}{2}} + 2 \left( \frac{\tau_y}{\tau_w} \right) \left( 1 - \frac{r}{R} \right) \quad (4.30)
$$

Equation 4.30 applies for the region defined by $r_{\text{critical}} \leq r \leq R$. At this point, however, we need to determine the value of $r_{\text{critical}}$. Since the shear stress at $r_{\text{critical}}$ must equal the yield stress, i.e., $\tau_z = \tau_y$, it is easy to then show from Equation 4.5 that $r_{\text{critical}} = R(\tau_y/\tau_w)$. For locations from the centerline to the critical radius, the velocity of the core is given by Equation 4.30 after setting $r/R = r_{\text{critical}}/R = \tau_y/\tau_w$. 

---

**Figure 4.6 Reduced average velocity versus wall shear stress.**
4.7.4 Tube flow of blood at low shear rates

At low shear rates RBC aggregates start to form due to the effect of plasma proteins like fibrinogen that make the RBCs stick together. This causes the apparent viscosity of blood to increase rapidly at low shear rates as we see in Figure 4.5. As these aggregates grow, their characteristic size can also become comparable to that of the tube diameter. The assumption of blood being a homogeneous fluid is then no longer correct. However, for practical purposes, blood flow in the body and through medical devices will be at significantly higher shear rates.

Therefore, we really need not concern ourselves with the limiting case of blood flow at these very low shear rates.

4.8 The effect of tube diameter at high shear rates

Tube flow of blood at high shear rates (i.e., $>100\, s^{-1}$) in tubes less than about 500 $\mu$m shows two anomalous effects that involve the tube diameter. These are the Fahraeus effect and the Fahraeus–Lindqvist effect (Barbee and Cokelet, 1971a,b; Gaehtgens, 1980; Pries et al., 1996).

4.8.1 The Fahraeus effect

To understand the effect of tube diameter on blood flow, consider the conceptual model illustrated in Figure 4.7. In this figure, blood flows from a larger vessel, such as an artery, into a smaller vessel after which the blood flows into a larger vessel as in a vein. We make the distinction between the hematocrit found in the feed to the smaller vessel, represented by $H_F$, the hematocrit in the vessel of interest $H_T$, and the discharge hematocrit $H_D$. All of these hematocrits are averaged over the entire cross-sectional area of the tube.

In tubes with diameters from about 10 to 500 $\mu$m, it is found that the tube hematocrit ($H_T$) is actually less than that of the discharge hematocrit (at steady state, $H_D = H_F$, to satisfy the mass balance on the RBCs). This is called the Fahraeus effect and $H_T/H_D$ reaches its minimal value of about 0.6–0.7 in tubes with a diameter of about 10–20 $\mu$m.

Pries et al. (1996) have shown that the experimental data for $H_T/H_D$ can be described by the following equation that depends on the tube diameter ($d$, microns) and the discharge hematocrit ($H_D$):

$$\frac{H_T}{H_D} = H_D + (1 - H_D)\left(1 + 1.7e^{-0.415d} - 0.6e^{-0.011d}\right)$$

(4.31)

To explain the Fahraeus effect, it has been shown both by in vivo and in vitro experiments that the RBCs do not distribute themselves evenly across the tube cross section. Instead, the RBCs tend to

![Figure 4.7 The Fahraeus effect.](9781498768719_C004.indd_170)
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accumulate along the tube axis forming, in a statistical sense, a thin cell-free layer along the tube wall. This is illustrated in Figure 4.8. Recall from Equation 4.7 that the fluid velocity is maximal along the tube axis. The axial accumulation of RBCs, in combination with the higher fluid velocity near the tube axis, maintains the RBC balance \((H_f = H_d)\) even though the hematocrit in the tube is reduced. The thin cell-free layer along the tube wall is called the plasma layer \((\delta)\). The thickness of the plasma layer depends on the tube diameter and the hematocrit and is typically on the order of several microns.

4.8.2 The Fahraeus–Lindqvist effect

Because of the Fahraeus effect, it is found that as the tube diameter decreases below about 500 µm, the apparent viscosity of blood also decreases, reaching a minimum value at a tube diameter of about 7 µm. Even though the shear rate is high enough for the blood to behave like a Newtonian fluid, the blood viscosity is no longer constant and just a material property of the blood, but also depends on the tube diameter. The decreased viscosity is a direct result of the decrease in the tube hematocrit and the presence of the cell-free plasma layer near the wall of the tube, which has the effect of reducing the flow resistance of the blood. This reduction in the viscosity of the blood as the tube diameter gets smaller is known as the Fahraeus–Lindqvist effect.

For the laminar flow of blood in tubes less than 500 µm, we can still use Equation 4.10 to express how the flow rate, i.e., \(Q\), depends on the pressure drop and the tube dimensions \(R\) and \(L\). However, we will need to replace the viscosity, i.e., \(\mu\), by its apparent value, i.e., \(\mu_{\text{apparent}}\), to account for the effect of the tube diameter. Hence, we can write Equation 4.10 as follows:

\[
Q = \frac{\pi R^4 \Delta P}{8 \mu_{\text{apparent}} L}
\]  (4.32)

Pries et al. (1996) have shown that the in vitro experimental data for the apparent viscosity of blood as a function of tube diameter \((d, \mu m)\) and the discharge hematocrit \((H_d)\) can be described by the following set of equations:

\[
\frac{\mu_{\text{apparent}}}{\mu_{\text{plasma}}} = 1 + \left(\eta_{0.45} - 1\right) \frac{(1-H_d)^c - 1}{(1-0.45)^c - 1}
\]  (4.33)
where $\eta_{0.45}$ is the ratio of the apparent viscosity of blood to plasma at a hematocrit of 0.45 and is given by

$$\eta_{0.45} = 220e^{-1.3d} + 3.2 - 2.44e^{-0.06d^{0.645}}$$

(4.34)

and the curve fitting parameter C is given by

$$C = \left(0.8 + e^{-0.05d}\right)\left(-1 + \frac{1}{1 + 10^{-11}d^{12}} + \frac{1}{1 + 10^{-11}d^{12}}\right)$$

(4.35)

For the in vivo case, Pries et al. (1996) provided the following expression for the apparent viscosity of blood as a function of the vessel diameter (d, $\mu$m) and the discharge hematocrit ($H_D$):

$$\frac{\mu_{\text{apparent}}}{\mu_{\text{plasma}}} = \left[1 + \left(\eta^*_{0.45} - 1\right) \left(\frac{1 - H_D}{1 - 0.45}\right)^C - 1 \left(\frac{d}{d-1.1}\right)^2\right] \left(\frac{d}{d-1.1}\right)^2$$

(4.36)

with $C$ given by Equation 4.35 and $\eta^*_{0.45}$ given by

$$\eta^*_{0.45} = 6e^{-0.085d} + 3.2 - 2.44e^{-0.06d^{0.645}}$$

(4.37)

The in vivo apparent viscosity of blood in vessels with diameters in the range of 4–30 $\mu$m is significantly higher than the in vitro apparent viscosity in tubes of the same diameter. This is likely due to the presence of macromolecules on the endothelial surface of blood vessels retarding the flow of plasma.

**Example 4.7**

Blood is flowing within a small hollow fiber module containing a total of 10,000 fibers. Each fiber has an internal diameter of 60 $\mu$m and a length of 20 cm. If the total flow rate of the blood is 10 mL min$^{-1}$, estimate the pressure drop in mmHg across the hollow fibers. Also calculate the tube hematocrit, $H_T$. Assume the discharge hematocrit of the blood is 0.40 and that the Newtonian viscosity of blood in large tubes is 3 cP. The plasma viscosity is 1.2 cP.

**Solution**

Since the hollow fiber diameter is less than 500 $\mu$m, we will need to take into account the Fahraeus and the Fahraeus–Lindqvist effects. First, we can calculate the tube hematocrit using Equation 4.31:

$$\frac{H_T}{0.40} = 0.40 + \left(1 - 0.40\right)\left(1 + 1.7e^{-0.415\times60} - 0.6e^{-0.011\times60}\right)$$

$$H_T = 0.326$$
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Next, we calculate the apparent viscosity of blood using Equations 4.33 through 4.35:

\[ C = (0.8 + e^{-0.075 \times 60}) \left( -1 + \frac{1}{1 + 10^{-1160^{0.12}}} \right) + \frac{1}{1 + 10^{-1160^{0.12}}} = -0.811 \]

\[ \eta_{0.45} = 220e^{-1.3 \times 60} + 3.2 - 2.44e^{-0.06 \times 60^{0.445}} = 2.148 \]

\[ \mu_{\text{apparent}} = 1.2cP = 1 + (2.148 - 1) \left( \frac{1 - 0.40}{1 - 0.45} \right) = 1.944 \]

\[ \mu_{\text{apparent}} = 1.944 \times 1.2cP = 2.333cP = 0.00233 \text{ Pas} \]

The flow rate for a given fiber is

\[ Q_{\text{fiber}} = 10 \text{ cm}^3 \text{ min}^{-1} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{1}{10,000 \text{ fibers}} = 1.667 \times 10^{-5} \text{ cm}^3 \text{ s}^{-1} \]

We can now rearrange Equation 4.32 and solve for the pressure drop across the hollow fiber as shown next:

\[ \Delta P = \frac{8 \mu_{\text{apparent}} L Q}{\pi R^3} = \frac{8 \times 0.00233 \text{ Pa s} \times 20 \text{ cm} \times 1.667 \times 10^{-5} \text{ cm}^3 \text{ s}^{-1}}{\pi \times (0.003 \text{ cm})^3} \]

\[ \Delta P = 24,422 \text{ Pa} \times \frac{1 \text{ atm}}{101,325 \text{ Pa}} \times \frac{760 \text{ mmHg}}{1 \text{ atm}} = 183.2 \text{ mmHg} \]

4.8.3 Marginal zone theory

The marginal zone theory proposed by Haynes (1960) may be used to characterize the Fahraeus–Lindqvist effect in the range from about 4–7 to 500 μm. Using this theory, it is possible to obtain an expression for the apparent viscosity in terms of the plasma layer thickness, tube diameter, and the hematocrit. Development of the marginal zone theory makes use of the relationships for a Newtonian fluid that we developed earlier. These are summarized in Table 4.3.

As shown in Figure 4.8, the blood flow within a tube or vessel is divided into two regions; a central core that contains the RBCs with a viscosity \( \mu_{\text{core}} \), and the cell-free marginal or plasma layer that consists only of plasma with a thickness of \( \delta \), and a viscosity equal to that of the plasma given by the symbol \( \mu_{\text{plasma}} \). Note that the tube hematocrit (i.e., averaged over the entire tube cross section) is related to the core hematocrit by the expression \( H_T = \left( 1 - \frac{\delta}{R} \right)^2 H_C \).

In each of these regions, the flow is considered to be Newtonian and Equation 4.6 applies to each. For the core region, we can then write

\[ \tau_{\text{fr}} = \left( \frac{P_0 - P_c}{2L} \right) = -\mu_{\text{core}} \frac{dv_{\text{core}}}{dr} \]

BC1: \( r = 0 \), \( \frac{dv_{\text{core}}}{dr} = 0 \) \hspace{1cm} (4.38)

BC2: \( r = R - \delta \), \( \tau_{\text{fr}}|_{\text{core}} = \tau_{\text{fr}}|_{\text{plasma}} \)
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The first boundary condition (BC1) expresses the fact that the axial velocity profile is symmetric about the axis of the tube and the velocity is also a maximum at the centerline of the tube. The second boundary condition (BC2) derives from the fact that the transport of momentum must be continuous across the interface between the core and the plasma layer.

For the plasma layer, the following equations apply:

\[ \tau_{rr} = \frac{(P_0 - P_L) r}{2L} = -\mu_{\text{plasma}} \frac{dv_z^{\text{plasma}}}{dr} \]

BC3: \( r = R - \delta, \quad v_z^{\text{core}} = v_z^{\text{plasma}} \)

BC4: \( r = R, \quad v_z^{\text{plasma}} = 0 \)  \hspace{1cm} (4.39)

Boundary condition three (BC3) states the requirement that the velocity in each region must be the same at their interface. The last boundary condition (BC4) simply requires that the axial velocity is zero at the tube wall.

Equations 4.38 and 4.39 can be readily integrated to give the following expressions for the axial velocity profiles in the core and plasma regions:

\[ v_z^{\text{plasma}}(r) = \frac{(P_0 - P_L) R^2}{4 \mu_{\text{plasma}} L} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \quad \text{for} \quad R - \delta \leq r \leq R \]  \hspace{1cm} (4.40)

\[ v_z^{\text{core}}(r) = \frac{(P_0 - P_L) R^2}{4 \mu_{\text{plasma}} L} \left\{ 1 - \left( \frac{R - \delta}{R} \right)^2 - \frac{\mu_{\text{plasma}}}{\mu_{\text{core}}} \left( \frac{r}{R} \right)^2 + \frac{\mu_{\text{plasma}}}{\mu_{\text{core}}} \left( \frac{R - \delta}{R} \right)^2 \right\} \quad \text{for} \quad 0 \leq r \leq R - \delta \]  \hspace{1cm} (4.41)

The plasma and core volumetric flow rates are given by the following equations:

\[ Q_{\text{plasma}} = 2\pi \int_{R - \delta}^{R} v_z^{\text{plasma}}(r)rdr \]

\[ Q_{\text{core}} = 2\pi \int_{0}^{R - \delta} v_z^{\text{core}}(r)rdr \]  \hspace{1cm} (4.42)

Integration of Equation 4.42 with the values of \( v_z^{\text{plasma}} \) and \( v_z^{\text{core}} \) from Equations 4.40 and 4.41 provides the following result for the volumetric flow rates of the plasma layer and the core:

\[ Q_{\text{plasma}} = \frac{\pi(P_0 - P_L) L}{8 \mu_{\text{plasma}}} \left[ R^2 - (R - \delta)^2 \right]^2 \]  \hspace{1cm} (4.43)

\[ Q_{\text{core}} = \frac{\pi(P_0 - P_L) R^2}{4 \mu_{\text{plasma}} L} \left[ (R - \delta)^2 - \left( 1 - \frac{\mu_{\text{plasma}}}{\mu_{\text{core}}} \right) \frac{(R - \delta)^4}{R^2} - \frac{\mu_{\text{plasma}}}{2 \mu_{\text{core}}} \frac{(R - \delta)^3}{2R^2} \right] \]  \hspace{1cm} (4.44)
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The total flow rate of blood within the tube is equal to the sum of the flow rates in the core and plasma regions. After adding Equations 4.43 and 4.44, we obtain the following expression for the total flow rate:

\[ Q = \frac{\pi R^4 (P_0 - P_L)}{8 \mu_{plasma} L} \left[ 1 - \left( 1 - \frac{\delta}{R} \right)^4 \left( 1 - \frac{\mu_{plasma}}{\mu_{core}} \right) \right] \]  

(4.45)

Comparing Equation 4.45 with Equation 4.32 allows one to develop a relationship for the apparent viscosity based on the marginal zone theory. We then arrive at the following expression for the apparent viscosity in terms of \( \delta, R, \mu_{core}, \) and \( \mu_{plasma} \):

\[ \mu_{apparent} = \frac{\mu_{plasma}}{1 - \left( 1 - \frac{\delta}{R} \right)^4 \left( 1 - \frac{\mu_{plasma}}{\mu_{core}} \right)} \]  

(4.46)

As \( \delta/R \to 0 \), then \( \mu_{apparent} \to \mu_{core} \to \mu \), which is the viscosity of blood in a large tube at high shear rates, i.e., a Newtonian fluid, as one would expect. We can use Equations 4.45 and 4.46 for blood flow calculations in tubes less than 500 \( \mu m \) in diameter provided that we know the thickness of the plasma layer, \( \delta \), and the viscosity of the core, \( \mu_{core} \). The next section shows how to determine the thickness of the plasma layer, \( \delta \), and the viscosity of the core, \( \mu_{core} \), using apparent viscosity data.

### 4.8.3.1 Using the marginal zone theory

If we have apparent viscosity data for blood at high shear rates in tubes of various diameters we can use Equation 4.46 to fit these data. First, we can use the binomial series to approximate the term \( (1 - r/R)^4 \) as follows:

\[ \left( 1 - \frac{\delta}{R} \right)^4 \approx 1 - \frac{4 \delta}{R} \]  

(4.47)

Substituting Equation 4.47 into Equation 4.46 and rearranging, we obtain

\[ \frac{1}{\mu_{apparent}} = \frac{1}{\mu_{core}} + \left[ \frac{4 \delta}{\mu_{core} \left( \frac{\mu_{plasma}}{\mu_{core}} - 1 \right)} \right] \frac{1}{R} \]  

(4.48)

Equation 4.48 shows us that a plot of \( 1/\mu_{apparent} \) versus \( 1/R \) should be linear with a y intercept equal to \( 1/\mu_{core} \) and a slope equal to the bracketed term. For a given plasma viscosity, the core viscosity can then be obtained from the y intercept and we can solve for the thickness of the plasma layer (\( \delta \)) from the slope. Example 4.8 shows how this is done.

#### Example 4.8

Find the values of \( \delta \) and \( \mu_{core} \) that represent the data (Bayliss, 1952; Gahtgens, 1980) for the apparent viscosity of blood as a function of tube diameter given in the paper by Gahtgens (1980) and summarized in the following table. Assume a plasma viscosity of 1.2 cP.
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<table>
<thead>
<tr>
<th>Tube Diameter, μm</th>
<th>μ&lt;sub&gt;apparent&lt;/sub&gt;, cP</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>1.89</td>
</tr>
<tr>
<td>30</td>
<td>2.07</td>
</tr>
<tr>
<td>40</td>
<td>2.31</td>
</tr>
<tr>
<td>60</td>
<td>2.46</td>
</tr>
<tr>
<td>80</td>
<td>2.70</td>
</tr>
<tr>
<td>100</td>
<td>2.79</td>
</tr>
<tr>
<td>130</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Solution

Figure 4.9 shows a plot of 1/μ<sub>apparent</sub> versus 1/R. The line shown in Figure 4.9 is the result of a linear regression that provided an intercept of b = 0.2986 cP<sup>-1</sup> and a slope of m = 2.8367 μm cP<sup>-1</sup> and a r² = 0.988. From Equation 4.48, we see that the core viscosity is equal to 1/b = μ<sub>core</sub> = 3.349 cP. From the slope m, we can solve for the value of the plasma layer thickness, i.e., δ:

\[
\delta = \frac{m \mu_{\text{core}}}{4\left(\frac{\mu_{\text{core}}}{\mu_{\text{plasma}}} - 1\right)} = \frac{2.8367 \mu\text{cP}^{-1} \times 3.349 \text{cP}}{4\left(\frac{3.349\text{cP}}{1.2\text{cP}} - 1\right)} = 1.326 \mu\text{m}
\]

Figure 4.9 Regression using Equation 4.48.

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4.9 Boundary layer theory

Describing the flow of a fluid near a surface is extremely important in a wide variety of engineering problems. Generally, the effect of the fluid viscosity is such that the fluid velocity changes from zero at the surface to the free stream value over a narrow region near the surface that is referred to as the boundary layer. It is the presence of this boundary layer that affects the rates of mass transfer and heat transfer between the surface and the fluid. Recall that the transport of something like momentum, mass, or energy is proportional to a driving force divided by a resistance. We will see that the boundary layer creates the resistance to the transport of momentum, mass, and energy. Interestingly, we will find that the thickness of the boundary layer is inversely proportional to the square root of the free stream velocity.

Analysis of these types of problems using boundary layer theory for relatively simple cases can provide a great deal of insight on how the flow of the fluid affects the transport of momentum, mass, and energy and lead to the rational development of correlations to describe their transport in more complex geometries and flow systems. We will use these correlations in later chapters in problems that involve the transport of mass across a bounding surface into a flowing fluid.

4.9.1 The flow near a wall that is set in motion

Consider the situation shown in Figure 4.10. A semi-infinite quantity of a viscous fluid is contacted from below by a flat and horizontal plate. For $t < 0$, the plate and the fluid are not moving. At $t = 0$, the plate is set in motion with a constant velocity ($V$) to the right, as shown in Figure 4.10. There are no...
pressure gradients or gravitational forces acting on the fluid, so the motion of the fluid is solely due to the momentum transferred from the plate to the fluid. The flow is also laminar, meaning there is no mixing of fluid elements in the y direction. The velocity in Cartesian coordinates will have components $v_x$, $v_y$, and $v_z$ and these will depend, in general, on x, y, z, and t. Since the plate does not move in the y direction, and we have no holes in the plate, $v_y$ is zero. Also, the fluid and plate are assumed to be infinite in the z direction, so $v_z$ is zero. The only component of the fluid velocity is therefore $v_x$.

Since the plate and the fluid are infinite in the x and z directions, $v_x$ can only depend on y, which makes sense since momentum is transported from the plate into the fluid in the y direction only. As time progresses, the x momentum of the plate is transported into the fluid in the y direction. In other words, for this laminar flow, the momentum diffuses in the y direction from a region of high velocity to a region of low velocity. This creates a velocity profile in the y direction that will depend on time.

We can arbitrarily define the boundary layer thickness ($\delta$) for this problem as the distance perpendicular to the plate surface where the fluid has just been set into motion, and this distance is defined as that point where the local velocity is equal to 1% of $V$. Our goal here is now to determine the velocity profile $v_x(y,t)$ and the boundary layer thickness, $\delta(t)$.

The concept of a shell balance can be used to analyze this problem. The shell balance is an important technique for developing mathematical models to describe the transport of such quantities as momentum, mass, and energy. The shell balance approach is conceptually easy to use and is based on the application of the generalized balance equation (Equation 1.8) to a given finite volume of interest.

In Figure 4.10, consider a small volume element of the fluid $\Delta x \Delta y W$, where W is the width of the plate in the z direction (normal to the page) and is assumed to be very large. Recall that momentum is (mass) $\times$ (velocity) and we can write the momentum per unit volume of the fluid as $\rho v_x$. The rate of accumulation of momentum within this volume element of the fluid is equal to $\rho \frac{\partial v_x}{\partial t} \Delta x \Delta y W$. This term has units of force and is also equal to the sum of the forces acting on this volume element of the fluid. The only forces acting on this volume element of fluid are the shear forces acting on the surfaces at y and y + $\Delta y$, i.e., $\tau_{yx}y$ and $\tau_{yx}(y+\Delta y)$. These terms also, respectively, represent the flux of x momentum in the y direction at y and y + $\Delta y$, respectively. Hence, our momentum shell balance on the volume element can be written as

$$\rho \frac{\partial v_x}{\partial t} \Delta x \Delta y W = \left( \tau_{yx}|_y - \tau_{yx}|_{y+\Delta y} \right) \Delta x W$$  (4.49)

Eliminating $\Delta x$ and W, and then dividing by $\Delta y$ and taking the limit as $\Delta y \rightarrow 0$, we obtain the following partial differential equation:

$$\rho \frac{\partial v_x}{\partial t} = - \frac{\partial \tau_{yx}}{\partial y}$$  (4.50)

This equation is valid for any fluid, Newtonian or non-Newtonian. For the special case of the Newtonian fluid, we can use Newton’s law of viscosity, i.e., Equation 4.2a and b, for $\tau_{yx}$, and obtain

$$\frac{\partial v_x}{\partial t} = - \nu \frac{\partial^2 v_x}{\partial y^2}$$  (4.51)
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where \( \nu \) is the *kinematic viscosity* and is defined as \( \mu/\rho \). The initial condition (IC) and boundary conditions (BC) are

\[
\begin{align*}
\text{IC:} & \quad t = 0, \nu_x = 0 \quad \text{for all values of } y \\
\text{BC1:} & \quad y = 0, \nu_x = V \quad \text{for all values of } t > 0 \\
\text{BC2:} & \quad y = \infty, \nu_x = 0 \quad \text{for all values of } t > 0
\end{align*}
\]

Equation 4.51 can easily be solved using the Laplace transform technique. Recall that the Laplace transform of a function \( f(t) \) is defined by the following equation:

\[
\mathcal{L}\{f(t)\} = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt
\]

Equation 4.54 can then be written as

\[
\frac{d^2\nu_x}{dy^2} - a^2 \nu_x = 0
\]

with \( a^2 = s/\nu \). Equation 4.56 is a homogeneous second order differential equation having the general solution

\[
\nu_x = C_1 e^{\frac{a^2}{2}y} + C_2 e^{-\frac{a^2}{2}y}
\]

The constants \( C_1 \) and \( C_2 \) can then be found from the transformed boundary conditions given by Equation 4.55. Using these boundary conditions we find that \( C_1 = 0 \) and \( C_2 = V/s \). Hence, our solution in the Laplace transform space is

\[
\frac{\nu_x}{V} = \frac{1}{s} e^{-\frac{a^2}{2}y} = \frac{1}{s} e^{-\frac{s}{\nu}\sqrt{y}}
\]
Table 4.4 Some Commonly Used Laplace Transforms

<table>
<thead>
<tr>
<th>ID</th>
<th>Function</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_1 f(t) + C_2 g(t)$</td>
<td>$C_1 \mathcal{L}(f(t)) + C_2 \mathcal{L}(g(t))$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{df(t)}{dt}$</td>
<td>$sF(s) - f(0^+)$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{\partial^2 f(x,t)}{\partial x^2}$</td>
<td>$\frac{s^2 F(x,s)}{c^2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\int_0^t f(\tau) d\tau$</td>
<td>$\frac{1}{s} \mathcal{L}(f(t))$</td>
</tr>
<tr>
<td>5</td>
<td>$f(\alpha t)$</td>
<td>$\frac{1}{\alpha} \mathcal{L}(f(s/\alpha))$</td>
</tr>
<tr>
<td>6</td>
<td>$e^{-\gamma t} f(t)$</td>
<td>$\mathcal{L}(f(t))$</td>
</tr>
<tr>
<td>7</td>
<td>$\int_0^1 f(t - \tau) g(\tau) d\tau$</td>
<td>$\mathcal{L}(f(t)g(t))$</td>
</tr>
<tr>
<td>8</td>
<td>$1$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>9</td>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{1}{(\pi t)^\nu}$</td>
<td>$\frac{1}{s^{\nu+1}}$</td>
</tr>
<tr>
<td>11</td>
<td>$-\frac{1}{2\pi^{1/2} t^\nu}$</td>
<td>$s^{1/2}$</td>
</tr>
<tr>
<td>12</td>
<td>$e^{\alpha t}$</td>
<td>$\frac{1}{s + \alpha}$</td>
</tr>
<tr>
<td>13</td>
<td>$\frac{e^{\alpha t} - e^{-\beta t}}{\alpha - \beta}$</td>
<td>$\frac{1}{(s + \alpha)(s + \beta)}$</td>
</tr>
<tr>
<td>14</td>
<td>$te^{-\alpha t}$</td>
<td>$\frac{1}{(s + \alpha)^2}$</td>
</tr>
<tr>
<td>15</td>
<td>$\frac{(\gamma - \beta)e^{-\alpha t} + (\alpha - \gamma)e^{-\beta t} + (\beta - \alpha)e^{-\gamma t}}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}$</td>
<td>$\frac{1}{(s + \alpha)(s + \beta)(s + \gamma)}$</td>
</tr>
<tr>
<td>16</td>
<td>$\frac{\frac{1}{2}e^{-\alpha t}}{t^\nu}$</td>
<td>$\frac{1}{s^{\nu+1}}$</td>
</tr>
<tr>
<td>17</td>
<td>$\frac{\alpha e^{-\alpha t} - \beta e^{-\beta t}}{\alpha - \beta}$</td>
<td>$\frac{s}{(s + \alpha)(s + \beta)}$</td>
</tr>
</tbody>
</table>

(Continued)
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Table 4.4 (Continued)  Some Commonly Used Laplace Transforms

<table>
<thead>
<tr>
<th>ID</th>
<th>Function</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>(\frac{(\beta - \gamma)\alpha e^{-\beta t} + (\gamma - \alpha)\beta e^{-\gamma t} + (\alpha - \beta)\gamma e^{-\alpha t}}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)})</td>
<td>(\frac{s}{(s + \alpha)(s + \beta)(s + \gamma)})</td>
</tr>
<tr>
<td>19</td>
<td>(\sin \alpha t)</td>
<td>(\frac{\alpha}{s^2 + \alpha^2})</td>
</tr>
<tr>
<td>20</td>
<td>(\cos \alpha t)</td>
<td>(\frac{s}{s^2 + \alpha^2})</td>
</tr>
<tr>
<td>21</td>
<td>(\sinh \alpha t)</td>
<td>(\frac{\alpha}{s^2 - \alpha^2})</td>
</tr>
<tr>
<td>22</td>
<td>(\cosh \alpha t)</td>
<td>(\frac{s}{s^2 - \alpha^2})</td>
</tr>
<tr>
<td>23</td>
<td>(\frac{x}{2(\pi t)^{1/2}} \exp \left(-\frac{x^2}{4at}\right))</td>
<td>(\exp(-x), \quad q = \left(\frac{s}{a}\right)^{1/2})</td>
</tr>
<tr>
<td>24</td>
<td>(\frac{a}{\pi t} \exp \left(-\frac{a^2}{4at}\right))</td>
<td>(\exp(-a^2), \quad q = \left(\frac{s}{a}\right)^{1/2})</td>
</tr>
<tr>
<td>25</td>
<td>(\text{erfc} \left(\frac{x}{2(\pi t)^{1/2}}\right))</td>
<td>(\exp(-\frac{x^2}{4st}), \quad q = \left(\frac{s}{a}\right)^{1/2})</td>
</tr>
<tr>
<td>26</td>
<td>(2 \left(\frac{at}{\pi}\right)^{1/2} \exp \left(-\frac{at}{2}\right) - x \text{erfc} \left(\frac{x}{2(\pi t)^{1/2}}\right))</td>
<td>(\exp(-\frac{x^2}{4st}), \quad q = \left(\frac{s}{a}\right)^{1/2})</td>
</tr>
<tr>
<td>27</td>
<td>(t + \frac{x^2}{2a} \exp \left(-\frac{x^2}{4at}\right) - \left(\frac{t}{\pi a}\right)^{1/2} \exp \left(-\frac{x^2}{4at}\right))</td>
<td>(\exp(-\frac{a^2}{4st}), \quad q = \left(\frac{s}{a}\right)^{1/2})</td>
</tr>
</tbody>
</table>


Now inverting Equation 4.58, we find the function \(v_x(t)\) whose Laplace transform is the above equation. From Table 4.4, we use transform pair 25 and then find that the inverse of Equation 4.58 provides the following solution for \(v_x(t)\):

\[
\frac{v_x(y,t)}{V} = \text{erfc} \left(\frac{y}{\sqrt{4vt}}\right) = 1 - \text{erf} \left(\frac{y}{\sqrt{4vt}}\right) \quad (4.59)
\]

This solution is in terms of a new function called the *error function* and the *complementary error function*, which are abbreviated as "erf" and "erfc," respectively. The error function is defined by the following equation:

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt \quad \text{and} \quad \text{erfc}(x) = 1 - \text{erf}(x) \quad (4.60)
\]
Note in Equation 4.60 that the integral of $e^{-x^2}$ cannot be obtained analytically. Since this integral is quite common in the solution of many engineering problems, this function has been tabulated in mathematical handbooks and mathematical software and can be treated as a known function, much like logarithmic and trigonometric functions.

Recall that we defined the boundary layer thickness as that distance $y$ from the surface of the plate where the velocity has decreased to a value of 1% of $V$. The complementary error function of $y/\sqrt{4Dt} = 1.821$ provides a value of $v/V$ that is equal to 0.01. Hence, we can define the boundary layer thickness, $\delta(t)$, as follows:

$$\delta(t) = 3.642\sqrt{vt} \approx 4\sqrt{vt} \quad (4.61)$$

The value of $\delta$ can also be interpreted as the distance to which momentum from the moving plate has penetrated into the fluid at time $t$.

**Example 4.9**

Calculate the boundary layer thickness $1 \text{ s}$ after the plate has started to move. Assume the fluid is water for which $\nu = 1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$.

**Solution**

Using Equation 4.61, we can calculate the thickness of the boundary layer as

$$\delta = 4\sqrt{10^{-6} \text{ m}^2 \text{ s}^{-1} \times 1 \text{ s}} = 0.004 \text{ m} = 4 \text{ mm}$$

### 4.9.2 Laminar flow of a fluid along a flat plate

Now consider the situation shown in Figure 4.11 for the steady laminar flow of a fluid along a flat plate. The plate is assumed to be semi-infinite and the fluid approaches the plate at a uniform
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velocity of \( V \) in the direction of the plate length, which is \( L \). The velocity in Cartesian coordinates will have components \( v_x, v_y, \) and \( v_z \) and these will depend, in general, on \( x, y, z \). However, since the plate is large in the \( z \) direction, there will be no \( v_z \) component of the velocity. Since the fluid velocity at the surface of the plate is zero, a boundary layer is formed near the surface of the plate, and within this boundary layer, \( v_x \) increases from zero at the plate surface to its free stream value of \( V \).

We can estimate the boundary layer thickness as follows (Schlichting, 1979). Recall for the plate suddenly set into motion that from Equation 4.61, the boundary layer thickness, \( \delta \), is proportional to \( \sqrt{vt} \), where \( t \) is the time, since the plate was set into motion. For the situation shown in Figure 4.11, we can think of \( t \) as the time it takes to travel from the leading edge of the flat plate to a downstream location \( x \). Hence, we can replace \( t \) in Equation 4.61 with \( x/V \), and we obtain the following approximate relationship for how the boundary layer thickness, \( \delta(x) \), changes along the surface of the flat plate:

\[
\delta(x) = 4 \frac{\sqrt{v_x}}{V} \tag{4.62}
\]

The growth of the boundary layer along the surface of the plate will also cause the fluid flow to be displaced in the \( y \) direction and this means, that in addition to the \( v_x \) component of the velocity, there will also be a \( v_y \) component. Both of these velocity components will depend on \( x \) and \( y \).

4.9.2.1 Approximate solution for laminar boundary flow over a flat plate

In the following discussion, we will obtain an approximate solution for the boundary layer flow over a flat plate. The approximate solution is very close to the exact solution. Details on exact solutions to the boundary layer equations may be found in Schlichting (1979).

Consider the shell volume shown in Figure 4.11 and located from \( x \) to \( x + \Delta x \) and from \( y = 0 \) to \( y = \delta(x) \). We first perform a steady-state mass balance on this shell volume that is given by

\[
\oint_{0}^{\delta(x)} \left[ W \rho v_x \right] - W \rho v_x \int_{k+\Delta x}^{x} dy - \rho v_y \int_{y=0}^{y=\delta(x)} W \Delta x = 0 \tag{4.63}
\]

The integral term in Equation 4.63 is the net rate (i.e., In–Out) at which mass is being added to the shell volume. The second term accounts for the loss of mass from the shell volume at the top of the boundary layer due to flow in the \( y \) direction. Eliminating \( \rho \) and \( W \), and dividing by \( \Delta x \), followed by taking the limit as \( \Delta x \to 0 \), provides the following equation* for \( v_y \mid_{y=\delta(x)} \):

\[
v_y \bigg|_{y=\delta(x)} = - \int_{0}^{\delta(x)} \frac{\partial v_x}{\partial x} dy = - \frac{d}{dx} \int_{0}^{\delta(x)} v_x dy + V \frac{d\delta(x)}{dx} \tag{4.64}
\]

* Here we have used Leibnitz’s rule, i.e.,

\[
\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t) dx = \int_{a(t)}^{b(t)} \frac{df}{dt} + f[b(t), t] \frac{db(t)}{dt} - f[a(t), t] \frac{da(t)}{dt}
\]
In a similar manner, we can write an x momentum balance on the shell volume as

$$\int_{0}^{\delta(x)} \left[ W\rho v_x v_x |_{k+\Delta k} - W\rho v_x v_x |_{k} \right] dy - \rho v_x |_{h=\delta(x)} V \Delta x + W \Delta x \tau_{yx} |_{y=0} = 0 \quad (4.65)$$

The first term from the left in Equation 4.65 represents the net rate at which x momentum is being added to the shell volume. The second term represents the rate at which x momentum per volume of fluid ($\rho V$) at the top of the boundary layer is being lost due to the flow of fluid out of the boundary layer in the y direction. The last term in this equation represents the loss of momentum as a result of the shear stress generated by the fluid at the surface of the plate.

After eliminating W and dividing by $\Delta x$, taking the limit as $\Delta x \to 0$, using Leibnitz’s rule* and Equation 4.64 to eliminate $v_y$, we can write Equation 4.65 as

$$-\tau_{yx} |_{y=0} = \frac{d}{dx} \left( \int_{0}^{\delta(x)} \rho (V - v_x) v_x dy \right) \quad (4.66)$$

For a Newtonian fluid, we can use Equation 4.2a and b and obtain

$$\mu \frac{\partial v_x}{\partial y} |_{y=0} = \frac{d}{dx} \left( \int_{0}^{\delta(x)} \rho (V - v_x) v_x dy \right) \quad (4.67)$$

Equation 4.67 is also known as the von Karman momentum balance equation and forms the basis for obtaining an approximate solution to the boundary layer flow over a flat plate. To obtain an approximate solution, we first need to approximate the shape of the velocity profile within the boundary layer, i.e., $v_x(x,y)$. The simplest function that reasonably approximates the shape of the velocity profile is a simple cubic equation:

$$v_x(y) = a + by + cy^2 + dy^3 \quad (4.68)$$

This velocity profile also has to satisfy the following boundary conditions:

BC1: $y = 0$, $v_x = 0$

BC2: $y = \delta(x)$, $v_x = V$

BC3: $y = \delta(x)$, $\frac{\partial v_x}{\partial y} = 0$

BC4: $y = 0$, $\frac{\partial^2 v_x}{\partial y^2} = 0 \quad (4.69)$

* Here we have used Leibnitz’s rule, i.e.,

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t) dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f[b(t),t] \frac{db}{dt} - f[a(t),t] \frac{da}{dt}$$
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The first boundary condition is referred to as the “no-slip” boundary condition, which requires that the velocity of the fluid at the surface of the plate be the same as the velocity of the plate, which, in this case, is zero. The last boundary condition expresses the fact that the stress at the surface of the plate only depends on x and not on y. Boundary conditions 2 and 3 state that beyond the boundary layer the velocity is constant and equal to the free stream value, V.

When the above boundary conditions are imposed on Equation 4.68, the following expression is obtained for the velocity profile within the boundary layer:

\[
\frac{v_x(x,y)}{V} = \frac{3}{2} \left( \frac{y}{\delta(x)} \right) - \frac{1}{2} \left( \frac{y}{\delta(x)} \right)^{3/2}
\]  

(4.70)

Equation 4.70 shows that \(v_x\) depends on \(\delta(x)\), which is still not known. However, we can insert this equation into the von Karman momentum balance equation (Equation 4.67) and after some simplification we obtain

\[
\frac{d}{dx} \left( \frac{1}{\delta(x)} \right) = \frac{140}{13} \left( \frac{\mu}{\rho V} \right)
\]

(4.71)

with the condition that at \(x = 0\), \(\delta = 0\). Integration of this equation results in the following expression for the boundary layer thickness, \(\delta(x)\):

\[
\delta(x) = 4.64 \sqrt{\frac{v_x}{V}}
\]

(4.72)

This equation is very similar to the approximate relationship we got earlier, i.e., Equation 4.62. We can also rearrange Equation 4.72 and express the boundary layer thickness relative to the downstream location \(x\) as

\[
\frac{\delta(x)}{x} = \frac{4.64}{\sqrt{\frac{\rho V x}{\mu}}} = 4.64 \frac{Re_x^{-1/2}}{\rho V x}
\]

(4.73)

In this equation, \(Re_x = \rho V x/\mu\) is defined as the local value (i.e., at location \(x\)) of the Reynolds number. The Reynolds number is a very important dimensionless number in the field of fluid mechanics. We see that the Reynolds number represents the ratio of the inertial forces \((\rho V x V L^2 = \rho V^2 L^2)\) acting on the fluid to the viscous forces \((\mu(V/L) x L^2 = \mu VL)\) acting on the fluid, where \(L\) is a characteristic dimension. A high Reynolds indicates that the inertial forces dominate the viscous forces. On the other hand, a low Reynolds number means viscous forces are much larger than the inertial forces. The Reynolds number also provides insight into when the fluid transitions from uniform laminar flow to turbulent flow. This critical Reynolds number for transition from laminar to turbulent flow depends on the geometry of the flow being considered. For example, for boundary layer flow over the flat plate, experiments show that the flow is laminar provided \(Re_x < 300,000\), whereas for flow in a cylindrical tube of diameter \(d\), the flow is laminar if the \(Re_d < \sim 2300\) and can transition to turbulent...
flow at a $Re > \approx 4000$. In this case of flow in a cylindrical tube, note that the characteristic dimension is the diameter of the cylindrical tube ($d$). For flow over the flat plate of length $L$, the Reynolds number is $Re_L = \rho VL/\mu$.

The approximate velocity profile within the boundary layer for laminar flow on a flat plate is then given by the combination of Equations 4.70 and 4.72 as shown here:

$$\frac{v_x(x,y)}{V} = 0.3233 \left( \frac{y}{\sqrt{Vx}} \right) - 0.005 \left( \frac{y}{\sqrt{Vx}} \right)^3$$  \hspace{1cm} (4.74)

Note that $v_x(x,y)/V$ depends on the single dimensionless parameter $\eta$, where $\eta = y\sqrt{V/\nu}$.

Equation 4.74 provides an excellent representation of the actual measured velocity profile for laminar boundary flow over a flat plate (see Problem 4.27 at the end of this chapter).

We can also use Equation 4.64 to calculate the $y$ component of the velocity at the outer edge of the boundary layer, i.e., $v_y|_{y=\delta(x)}$, using Equation 4.74 for $v_x(x,y)$ and Equation 4.72 for $\delta(x)$:

$$v_y|_{y=\delta(x)} = 0.87V \sqrt{\frac{\nu}{\mu}} = 0.87V \frac{\mu}{\rho Vx} = 0.87V \frac{1}{\sqrt{Re_x}}$$  \hspace{1cm} (4.75)

The exact value of $v_y|_{y=\delta(x)}$ given by Schlichting (1979) has the constant in Equation 4.75 as 0.8604 instead of the value given by our approximate solution of 0.87. Equation 4.75 shows that at the outer edge of the boundary layer there is an outward flow of fluid in the $y$ direction that is caused by the growth of the boundary layer thickness in the direction of flow, i.e., $\delta(x)$, as given by Equation 4.72.

With the velocity profile given by Equation 4.74, we can also calculate the drag force exerted by the fluid on the plate. For a plate of length $L$ and width $W$, the force that acts on the surface of both sides of the plate in the positive $x$ direction is given by

$$F_x = 2 \int_0^W \left( \frac{\partial v_x}{\partial y} \right)_{y=0} \mathrm{d}x \mathrm{d}z = 1.293\sqrt{\rho \mu LW^2V^3}$$  \hspace{1cm} (4.76)

The exact solution, as well as experimental data for the drag force on the flat plate, is about 3% greater than that predicted by the approximate solution to the flat plate boundary layer problem given by Equation 4.76. The constant in Equation 4.76 being 1.328 for the exact solution. Hence, we see that this approximate solution to the flat plate boundary layer problem is quite good and not only predicting the $x$ and $y$ velocity profiles but also the drag as well.

We can also calculate the power needed to overcome the drag force. Recall that power is defined as (force $\times$ velocity), so after multiplying Equation 4.76 by $V$, the power is given by

$$P = VF_x = 1.293\sqrt{\rho \mu LW^2V^3}$$  \hspace{1cm} (4.77)
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The friction factor \( f \) is defined as the ratio of the shear stress at the wall and the kinetic energy per volume of the fluid based on the free stream velocity. The local value of the friction factor is then given by

\[
 f = \frac{\tau_w}{\frac{1}{2} \rho V^2} = \frac{\frac{\mu}{\delta}}{\frac{1}{2} \rho V^2} = \frac{3 \mu}{\rho V\delta(x)} = \frac{0.646}{\sqrt{Re}}
\]  
(4.78)

For a plate of length \( L \) and width \( W \), the average value of the friction factor \( f \) is given by integrating Equation 4.78 as shown here:

\[
f = \frac{\int_0^W \int_0^L \frac{0.646}{\sqrt{Re}} \, dx \, dz}{WL} = \frac{1.293 \frac{1}{\sqrt{Re_L}}}{W L}
\]  
(4.79)

where \( Re_L = \rho VL/\mu \). The definition of the friction factor given by Equation 4.78 means that the force acting on both sides of the plate (i.e., \( F_x \)) is given by the product of the kinetic energy per volume of the fluid \((1/2)\rho V^2\), the area of the flat plate \((2LW)\), and the friction factor \( f \) and is given by the next equation:

\[
F_x = \frac{1}{2} \rho V^2 \times \text{(Area)} \times f = \frac{1}{2} \rho V^2 \times (2LW) \times 1.293 \frac{1}{\sqrt{Re}} = 1.293 \sqrt{\rho \mu LW^2 V^3}
\]  
(4.80)

This result for \( F_x \) is, as expected, the same as that given by Equation 4.76. As shown by Equations 4.79 and 4.80, the friction factor is a convenient method for finding the force acting on a surface as a result of fluid motion. When the flow is across or over an object, the friction factor is also known as the drag coefficient \( f = \frac{C_D}{f} \) since it allows for the calculation of the drag force exerted on the object by the flowing fluid.

Example 4.10

Water \((\mu = 0.001 \text{ Pa s}, \rho = 1000 \text{ kg m}^{-3})\) is flowing over a flat plate at a speed of 0.1 \text{ m s}^{-1}. Over what length of the plate (cm) is the boundary layer flow laminar? What is the boundary layer thickness (cm) at the location where the flow becomes turbulent? If the plate has a width of 25 cm, what is the drag force (N) acting on the flat plate over this length?

Solution

The flow will transition to turbulent flow once the Reynolds number reaches 300,000. Letting this distance be where \( x = L \), the Reynolds number can be solved for the value of \( L \) as

\[
L = \frac{Re_L \mu}{\rho V} = \frac{300,000 \times 0.001 \text{ m}^2}{\rho \mu V} \times \frac{\text{kg m}}{s^2 \text{N}} = \frac{300,000 \times 0.001}{1000 \text{ kg m}^{-3} \times 0.1 \text{ m}} = 3.0 \text{ m} = 300 \text{ cm}
\]
From Equation 4.72, the thickness of the boundary layer at this location (i.e., L) can be found:

\[
\delta(L) = 4.64 \sqrt{\frac{\mu L}{\rho V}} = 4.64 \sqrt{\frac{0.001 \frac{Ns}{m^2} \times 3.0 \frac{m}{s} \times \frac{kgm}{s^2} N}{1000 \frac{kg}{m^2} \times 0.1 \frac{m}{s}}} = 0.0254 \text{ m} = 2.54 \text{ cm}
\]

The drag force acting on the plate can be found from Equation 4.76:

\[
F_{\text{drag}} = 1.293 \sqrt{1000 \frac{kg}{m^2} \times 0.001 \frac{Ns}{m^2} \times 3.0 \frac{m}{s} \times 0.25^2 \frac{m^2}{s^2} \times 0.1 \frac{m^3}{s^3} \times 1 \frac{Ns^2}{kgm}} = 0.018 \text{ N}
\]

**Example 4.11**

Consider a manta ray swimming along in the ocean at 0.75 m s⁻¹. Assuming the ray approximates a rectangular shape with a length of 0.30 m and a width of 0.75 m, estimate how much power the ray is expending to move through the water as a result of the drag force of the water on its surface? Assume the density of the water is 1000 kg m⁻³ and the viscosity of the water is 1 cP = 0.001 Pa s = 0.001.

**Solution**

Assume the geometry of the manta ray approximates that of a flat plate, and we can assume that we have laminar flow of the seawater over a flat plate. Using Equation 4.77 we can calculate the power requirement as follows:

\[
P_k = \frac{1.293}{1000 \frac{kgm}{s} \times 0.001 \frac{Ns}{kgm^{-1}s^{-1}} \times 0.3 \frac{m}{s} \times 0.75 \frac{m^2}{s^2} \times 0.75 \frac{m^5}{s^5}} = 0.26 \text{ kgm}^2 \text{s}^{-2} \text{ms}^{-1} = 0.26 \text{ J} \text{s}^{-1} = 0.26 \text{ W}
\]

As shown here, the Reynolds number calculation shows that the flow is laminar over the surface of the manta ray since it is less than 300,000 at \( x = L = 0.30 \text{ m} \):

\[
Re_L = \frac{1000 \frac{kgm}{s} \times 0.75 \frac{ms}{s} \times 0.30 \frac{m}{s}}{0.001 \frac{kgm}{s^2} \text{s}^{-1}} = 225,000
\]

### 4.10 Generalized mechanical energy balance equation

Equation 4.10 provides a useful relationship for describing the laminar flow of a liquid in a cylindrical tube of constant cross section. Oftentimes, we will need a more generalized relationship that can handle turbulent flow and account for not only the effect of the pressure drop on fluid flow, but also the effect of changes in elevation, tube cross section, changes in fluid velocity, sudden contractions or expansions, pumps, as well as the effect of fittings such as valves. This general relationship for the flow of a fluid is called the *Bernoulli equation* and is valid for laminar or turbulent flow.
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The Bernoulli equation accounts for the effect of changes in fluid pressure, potential energy, and kinetic energy on the flow of the fluid. It also accounts for the energy added to the fluid by pumps, the energy removed by turbines, and accounts for energy losses due to a variety of frictional effects.

For steady-state flow of an incompressible fluid (density $\rho = \text{constant}$) from inlet station “1” to exit station “2” (see Figure 2.1), the Bernoulli equation can be written as follows (McCabe et al., 1985; Bird et al., 2002):

$$\frac{P_1}{\rho} + gZ_1 + \frac{\alpha_1 V_1^2}{2} + W_{\text{device}} = \frac{P_2}{\rho} + gZ_2 + \frac{\alpha_2 V_2^2}{2} + h_{\text{friction}} \quad (4.81)$$

In this equation:
- $P$ represents the pressure
- $Z$ is the elevation relative to a reference plane
- $V$ is the average fluid velocity

Gravitational acceleration is represented by $g$ and in SI units is equal to $9.8 \text{ m s}^{-2}$. The $\alpha$'s are kinetic energy correction terms that account for the shape of the velocity profile. For laminar flow in a cylindrical tube, i.e., for a $Re = \rho_d u_d \nu < 2300$, $\alpha = 2.0$, and for turbulent flow $\alpha = 1.0$. The work effect per unit mass of fluid is $W_{\text{device}}$. If $W_{\text{device}} > 0$, then work is done on the fluid, e.g., by a pump. The actual work required to achieve these changes in the fluid properties will be greater than $W_{\text{device}}$ due to frictional losses and mechanical inefficiencies within the pump. If $W_{\text{device}} < 0$, then the fluid produces work, e.g., by a turbine. The actual amount of work that is generated will be less than this value because of frictional losses and mechanical inefficiencies within the turbine. Other frictional effects between positions 1 and 2 due to the tube itself, contractions, expansions, and fittings are accounted for by the term $h_{\text{friction}}$.

The units of each term in Equation 4.81 must be consistent with one another and are expressed in terms of energy per unit mass of fluid. In SI units, each term in Equation 4.81 will have units of $\text{m}^2 \text{ s}^{-2}$. If these units of $\text{m}^2 \text{ s}^{-2}$ are multiplied by $\text{kg kg}^{-1}$, then as we see in the following equation, the units are the same as a $\text{J kg}^{-1}$:

$$\frac{\text{m}^2}{\text{s}^2} \times \text{kg kg}^{-1} = \text{kg m s}^{-2} \times \text{kg kg}^{-1} = \text{Nm kg}^{-1} = \text{J kg}^{-1} \quad (4.82)$$

In addition to the Bernoulli equation, we also need to write a mass balance on the fluid. At steady state, this simply says that the rate of mass leaving the system, or a region of interest at position 2, must equal the rate at which mass enters the system at position 1. For steady-state flow, this can be expressed by the mass conservation or continuity equation given here:

$$\dot{m} = (\rho VS)_1 = (\rho VS)_2 = \text{constant} \quad (4.83)$$

In this equation:
- $\dot{m}$ is the mass flow rate
- $S$ represents the cross-sectional area of the tube
The frictional effects in Equation 4.81 are described by

\[
\frac{h_{\text{friction}}}{f_i L_i V_i^2} = \frac{4}{2d_i} \sum_i \frac{V_i^2}{2} + \sum_j K_{\text{fitting}}
\]

(4.84)

In this equation, \(f_i\) is the friction factor and accounts for the loss in fluid energy per unit mass of fluid in tube segment \(i\) of length \(L_i\) and diameter \(d_i\). \(V_i\) represents the average velocity,

\[
V_i = \frac{\text{Volumetric flowrate}}{\pi d_i^2 / 4}, \quad \text{within tube section.}
\]

The friction factor \(f\), also known as the Fanning friction factor, is defined as the ratio of the wall shear stress \((\tau_w)\) and the average kinetic energy per unit mass of fluid, also called the velocity head, i.e., \(pV^2/2\). For laminar flow, it is easy to show from Equation 4.10 that \(f = 16/\text{Re}\). For turbulent flow \((\text{Re} > \sim 2300)\) in smooth tubes, the friction factor may be evaluated from either of the following equations (McCabe et al., 1985; Bird et al., 2002):

\[
\frac{1}{\sqrt{f}} = 4.07 \log \left( \frac{\text{Re} \sqrt{f}}{4} \right) - 0.60
\]

(4.85)

The second equation is valid up to a \(\text{Re} = 100,000\) and is more convenient to use since \(f\) is explicit in \(\text{Re}\). If the flow is turbulent, and the walls are not smooth, then the friction factor charts in McCabe et al. or Bird et al. should be consulted (i.e., graphs of \(f\) versus \(\text{Re}\) with surface roughness as a parameter).

The term in the second summation of Equation 4.84, represented by \(K_{\text{fitting}}\), accounts for energy loss in the fluid due to tube contractions, expansions, or fittings such as valves. It is important to note that for the most part these fitting losses tend to be negligible for laminar flow of a fluid.

The average velocity in this second summation of Equation 4.84 is for the fluid just downstream of the contraction, expansion, or fitting. \(K_{\text{fitting}}\) for such items as valves is highly dependent on the valve’s degree of openness and on the type of valve used. It is generally best to consult the manufacturer’s literature for the particular valve being considered. As examples, a gate valve that is wide open has a \(K_{\text{fitting}}\) of about 0.2 and a value of about 6 when half open. A globe valve that is wide open may have a value as high as 10. Simple fittings, such as a tee, has a \(K_{\text{fitting}}\) value of 2.0, and a 90° elbow has a value of about 1.0.

For sudden contractions and expansions of the fluid, the following equations can be used to estimate \(K_{\text{fitting}}\). Once again, \(S\) refers to the cross-sectional area of the tube:

\[
K_{\text{contraction}} = 0.45 \left( 1 - \frac{S_{\text{downstream}}}{S_{\text{upstream}}} \right), \quad \text{sudden contraction, turbulent flow}
\]

(4.86)

\[
K_{\text{expansion}} = \left( \frac{S_{\text{downstream}}}{S_{\text{upstream}}} - 1 \right)^2, \quad \text{sudden expansion, turbulent flow}
\]
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If for an expansion $S_{\text{downstream}} \gg S_{\text{upstream}}$, then $K_{\text{expansion}} = 1$, and the velocity upstream of the expansion is used in Equation 4.84. In this case, the expansion means that we essentially lose all of the kinetic energy per mass of fluid (i.e., $V_{\text{upstream}}^2/2$) as it goes through the expansion.

4.10.1 The hydraulic diameter

If the tube through which a fluid flows is not circular, then the hydraulic diameter can be used in the above calculations. The hydraulic diameter for flow in tubes of noncircular cross section is defined as four times the hydraulic radius (i.e., $4 \times r_H$). The hydraulic radius is defined as the ratio of the cross-sectional area of the flow channel to the wetted perimeter of the tube. The factor of four is needed so that $d_H = d$ for a circular tube:

$$d_H = 4 \times r_H = \frac{4 \times \text{(Cross sectional area)}}{\text{(Wetted perimeter)}} \quad (4.87)$$

The hydraulic diameter is then used in the calculation of the Reynolds number, i.e., $Re = \rho V d_H / \mu$ and in Equation 4.84. However, the average velocity ($V$) is still defined as the volumetric flow rate ($Q$) divided by the cross-sectional area normal to the flow. It is also important to note that the hydraulic diameter should only be used for turbulent flow.

For flow in a cylindrical annulus with an inner tube of outer diameter $d_i$ and an outer tube with inner diameter $d_o$, the hydraulic diameter is found from Equation 4.87 to be $(d_o - d_i)$. For flow between two parallel plates of width, $W$, separated by a distance, $H$, and completely filled with fluid, the hydraulic diameter is equal to $2WH/(W + H)$. Note that for a very wide rectangular duct where $W \gg H$, the hydraulic diameter for this slit flow is $2H$, or twice the separation of the plates.

The following examples illustrate the use of the Bernoulli equation and the above relationships.

Example 4.12

Consider the flow of a fluid through an orifice located at the bottom of a tank. The cross-sectional area of the tank is $S_1$ and the area of the orifice located at the bottom surface of the tank is $S_2$. The fluid stream leaving through the orifice will tend to contract (called the vena contracta) such that at a short distance downstream of the orifice its cross-sectional area will be about 0.64 times that of the area of the orifice or $S_{\text{vena contracta}} = 0.64 S_2$. If the pressure in the tank is maintained at $P_1$, and if the fluid leaving through the orifice is at atmospheric pressure, develop an expression for the average fluid velocity at the vena contracta, assuming the height of the fluid in the tank is given by $h$.

Solution

First, we write the Bernoulli equation (4.81) for this situation, where the index 1 denotes the surface of the fluid within the tank. Note that there are no work devices and we ignore any frictional effects and assume the flow leaving the tank through the orifice is turbulent. We also set $g(Z_1 - Z_2) = g h$. Hence, for this situation we can write the Bernoulli equation as

$$\frac{P_1}{\rho} + gh + \frac{V_1^2}{2} = \frac{P_{\text{vena contracta}}}{\rho} + \frac{V_{\text{vena contracta}}^2}{2}$$

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Since $P_{\text{vena contracta}}$ is the same as the atmospheric pressure ($P_{\text{atm}}$), this equation can be rewritten as

$$V_{\text{vena contracta}}^2 = V_1^2 + 2\left(\frac{P_1 - P_{\text{atm}}}{\rho}\right) + 2gh$$

Mass conservation given by Equation 4.83 allows for the calculation of the velocity of the fluid surface within the tank in terms of the velocity at the vena contracta, i.e.,

$$V_1 = V_{\text{vena contracta}} \frac{S_{\text{vena contracta}}}{S_1}$$

Using the previous equation to eliminate $V_1$ in the previous equation for $V_{\text{vena contracta}}$ allows for the calculation of the average velocity of the fluid in the vena contracta as

$$V_{\text{vena contracta}} = \left[2\left(\frac{P_1 - P_{\text{atm}}}{\rho}\right) + 2gh\right]^{1/2}$$

For the flow of a fluid from a tank through an orifice, we usually have that $S_1 \gg S_{\text{vena contracta}}$. Hence, in this case

$$V_{\text{vena contracta}} = \sqrt{2\left(\frac{P_1 - P_{\text{atm}}}{\rho}\right) + 2gh}$$

And if the tank is open to the atmosphere, then $P_1 = P_{\text{atm}}$ and we obtain what is known as the Torricelli equation:

$$V_{\text{vena contracta}} = \sqrt{2gh}$$

Based on this result, the volumetric flow rate of the fluid, i.e., $Q = S_{\text{vena contracta}} \times V_{\text{vena contracta}}$, is given by the next result:

$$Q = S_{\text{vena contracta}} \sqrt{2gh} \approx 0.64 S_{\text{orifice}} \sqrt{2gh}$$

If the flow through the orifice is driven by the pressure difference and not by the potential energy of the fluid, i.e., if $\left(\frac{P_1 - P_{\text{atm}}}{\rho}\right) \gg 2gh$, then we have

$$V_{\text{vena contracta}} = \sqrt{\frac{2(P_1 - P_{\text{atm}})}{\rho}}$$

And in this case, $Q$ is given by

$$Q = S_{\text{vena contracta}} \sqrt{\frac{2(P_1 - P_{\text{atm}})}{\rho}} \approx 0.64 S_{\text{orifice}} \sqrt{\frac{2(P_1 - P_{\text{atm}})}{\rho}}$$  \hspace{1cm} (A)
Example 4.13

Consider a patient being evaluated for stenosis (narrowing) of their aortic valve. Catheterization of the heart gave a cardiac output of 5000 mL min\(^{-1}\), a mean systolic pressure drop across the aortic valve of 50 mmHg, a heart rate of 80 beats min\(^{-1}\), and a systolic ejection period of 0.33 s. From these data, estimate the cross-sectional area of the aortic valve.

Solution

We can use Equation A developed in the previous example to determine the cross-sectional area of the aortic valve (\(S_{\text{orifice}} = S_{\text{aortic valve}}\)) from these data. From Table 4.1, the density of blood is 1.056 g cm\(^{-3}\) = 1056 kg m\(^{-3}\). From the cardiac output and the number of heart beats per minute, we find that each beat of the heart moves a volume of blood given by:

\[
\frac{5000 \text{ cm}^3}{\text{min}} \times \frac{1 \text{ min}}{80 \text{ beats}} = 62.5 \text{ cm}^3 \text{ beat}^{-1}
\]

If the aortic valve remains open for 0.33 seconds for each beat, then each beat of the heart gives a flow rate of blood through the valve of:

\[
\frac{62.5 \text{ cm}^3 \text{ beat}^{-1}}{0.33 \text{ s}} = 189.4 \text{ cm}^3 \text{ s}^{-1}
\]

Solving Equation A for the area of the aortic valve:

\[
S_{\text{aortic valve}} = \frac{Q}{2 \Delta P_{\text{systolic}}} = \frac{0.64 \times \frac{189.4 \text{ cm}^3}{\text{s}}}{\sqrt{\frac{2 \times 50 \text{ mmHg}}{1056 \text{ kg/m}^3}} 	imes \frac{101,325 \text{ Pa}}{760 \text{ mmHg}} \times \frac{\text{N}}{\text{m}^2 \text{Pa}} \times \frac{\text{kg m}}{\text{N s}^2} \times \frac{10^4 \text{ cm}^2}{\text{m}^2}}
\]

\[
S_{\text{aortic valve}} = 0.83 \text{ cm}^2
\]

We can also calculate the Reynolds number at the vena contracta of the blood flowing out of the aortic valve. At the vena contracta, the cross-sectional area is 0.64 \(\times\) 0.83 cm\(^2\) = 0.53 cm\(^2\), which gives a diameter for the vena contracta of 0.82 cm. The average velocity of the blood at the vena contracta is given by:

\[
V = \frac{189.4 \text{ cm}^3}{\text{s}} \times \frac{0.64 \times 0.83 \text{ cm}^2}{1} = 356.6 \text{ cm/s}
\]

From Table 4.1, we have that the viscosity of blood is 3 cP. Therefore, we can calculate the Reynolds number as:

\[
\text{Re} = \frac{\rho V d_{\text{vena contracta}}}{\mu_{\text{blood}}} = \frac{1.056 \frac{\text{g}}{\text{cm}^3} \times 356.6 \frac{\text{cm}}{\text{s}} \times 0.82 \text{ cm}}{3 \text{ cP} \times 0.01 \frac{\text{g}}{\text{cP cm s}}} = 10,293
\]

Hence, we conclude that the flow of the blood through the aortic valve under these conditions is turbulent and that the assumptions used to calculate the aortic valve cross-sectional area that were based on the results of the previous example are valid.
Example 4.14

A Pitot tube (see Figure 4.12) is a simple device for measuring the velocity of a fluid that is passing nearby. This device is used to measure the airspeed on airplanes and can be inserted into pipes to measure the velocity of the flowing fluid. As shown in Figure 4.12, the Pitot tube is aligned with the axis of the flow. At the tip of this tube (1) there is a small opening into an inner tube that leads to a pressure measuring system. In this case, the pressure measuring system is a simple manometer that contains a denser fluid such as mercury with density \( \rho_m \). The fluid that impinges at the tip of the Pitot tube (1) stagnates or has a zero velocity (i.e., \( V_1 = 0 \)) and its kinetic energy is converted according to the Bernoulli equation into what is called the stagnation pressure, \( P_1 \). Surrounding or coaxial to this tube is another tube and along the sides of this tube are small holes that are also connected to the pressure measuring system. These small holes along the side of this outer tube sense the static pressure (\( P_2 \)) of the surrounding fluid that passes by at a velocity of \( V_2 \). From this information, develop an expression for the velocity of the fluid flowing near a Pitot tube in terms of the pressure difference (\( P_1 - P_2 \)).

Also, write this expression for the velocity in terms of the manometer reading (h) for a manometer fluid of density \( \rho_m \).

Solution

For the fluid flowing past the Pitot tube, there is no change in elevation. Also, since in this case we are measuring the local velocity of the fluid and not the average velocity, there is no kinetic energy correction term; hence \( \alpha \) is equal to one. If we also neglect any work and frictional effects, the Bernoulli equation (4.81) can be written between the stagnation point (1) (note \( V_1 = 0 \)) and the static holes (2) as

\[
\frac{P_1}{\rho} = \frac{P_2}{\rho} + \frac{V_2^2}{2}
\]

Note that this equation says that the stagnation pressure (\( P_1 \)) is equal to the static pressure (\( P_2 \)) plus the dynamic pressure, or the velocity head of the fluid, i.e., \( \frac{1}{2} \rho V^2 \); hence \( P_1 = P_2 + \frac{1}{2} \rho V^2 \).

This equation can then be rearranged and solved to give the velocity of the fluid passing near the static holes along the sides of the Pitot tube:

\[
V_2 = \sqrt{\frac{2(P_1 - P_2)}{\rho}}
\]

Figure 4.12 The Pitot tube.
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If a manometer is used as shown in Figure 4.12 to measure the pressure difference \((P_1 - P_2)\), then the Bernoulli equation can be used to express this pressure difference in terms of the difference in height \((h)\) between the legs of the manometer fluid. At the surface of the upper leg of the manometer fluid the pressure is \(P_2\) and the velocity of this surface is equal to zero. Similarly, at the surface of the lower leg of the manometer fluid, the pressure is \(P_1\) and the velocity of this surface is also zero. Neglecting work and frictional effects, the Bernoulli equation for this situation becomes

\[
\frac{P_1}{\rho_m} + gZ_1 = \frac{P_2}{\rho_m} + gZ_2
\]

Letting \(Z_2 - Z_1 = h\), this equation can be solved for \((P_1 - P_2)\) in terms of \(h\) for a manometer fluid of density \(\rho_m\). This result is also known as the manometer equation:

\[
(P_1 - P_2) = \rho_m gh
\]

Using this result, we can express the velocity for the Pitot tube of Figure 4.12 as

\[
V_z = \sqrt{\frac{2 \rho_m gh}{\rho}}
\]

**Example 4.15**

Farmer Jones needs to pump water from the Maumee River to irrigate his soybean fields. He figures that he will need 250 ft of 2 in. inside diameter steel pipe and four 90° elbows to bring the water from the river to his water tower where the water is then stored. The total change in elevation from the river surface to where the water flows into the top of the water tower is 100 ft. Farmer Jones wants the water flow rate in this system to be 100 gal min\(^{-1}\). Assuming the pump down by the river has an efficiency of 80%, estimate the required horsepower of the pump that is needed.

**Solution**

We assume that the storage tank is vented to the atmosphere and that the pipe enters at the top of the storage tank where the water flows into the storage tank. We also let location (1) be the surface of the river and location (2) is the plane at the exit of the pipe. With these assumptions, the pressure at the surface of the river is the same as that inside the storage tank; hence we have \(P_1 = P_2\). We also assume that the surface of the river adds no kinetic energy to the flow in the pipe which means that \(V_1 = 0\). With these assumptions, we can then write the Bernoulli equation (4.81) from the surface of the lake (1) to where the water leaves the pipe (2):

\[
gZ_1 + W_{device} = gZ_2 + \frac{\alpha_s V_i^2}{2} + h_{friction}
\]

We now use Equation 4.84 for \(h_{friction}\) to account for the resistance of the pipe of length \(L\) and diameter \(d\) and the fittings. The above equation then becomes

\[
W_{device} = g(Z_2 - Z_1) + \frac{\alpha_s V_i^2}{2} + \frac{4fLV_i^3}{2d} + \sum_j \frac{V_j^2}{2}K_{fitting}
\]
The second to the last term in the previous equation accounts for the loss of energy per unit mass as the water flows at velocity \( V_2 \) through the 2 in. pipe (d) that is 250 ft in length (L). The friction factor \( f \) in this term can be calculated after we find the Reynolds number and determine whether the flow in the pipe is laminar or turbulent. To find the Reynolds number, we first need to determine the velocity, \( V_2 \), in the pipe as

\[
V_2 = \frac{4Q}{\pi d^2} = \frac{4 \times 100 \text{ gal}}{\pi \times (2 \text{ in.})^2} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{0.13368 \text{ ft}^3}{\text{gal}} \times \frac{(12 \text{ in.})^3}{\text{ft}^3} \times \frac{(2.54 \text{ cm})^3}{\text{in.}^3} = 311.3 \text{ cm/s}
\]

Assuming that water has a density of 1 g cm\(^{-3}\) and a viscosity of 1 cP, we can now calculate the value of the Reynolds number:

\[
Re = \frac{\rho d V_2}{\mu} = \frac{1 \text{ g/cm} \times 2 \text{ in.} \times \frac{2.54 \text{ cm}}{\text{in}} \times \frac{311.3 \text{ cm}}{\text{s}}}{1 \text{ cP} \times \frac{1 \text{ g}}{\text{cm s}100 \text{ cP}}} = 158,140
\]

The flow in the pipe is therefore turbulent and \( \alpha_2 = 1.0 \). We can also use Equation 4.85 to find the friction factor, which is found to be equal to 0.0041.

Next, the contraction of the fluid as it enters the pipe from the river and the effect of the four elbows is calculated as

\[
\sum K_{\text{fitting}} = \frac{1}{2} V_2^2 \times (4 \times K_{\text{elbow}}) + 0.45 \left( 1 - \frac{S_{\text{pipe}}}{S_{\text{river}}} \right) \times \frac{1}{2} V_2^2 = 2.225 V_2^2
\]

This result is obtained when the value of \( K_{\text{elbow}} \) is set equal to 1. We also assume for the contraction from the river into the pipe that \( S_{\text{river}} \gg S_{\text{pipe}} \). The Bernoulli equation then becomes

\[
W_{\text{device}} = g (Z_2 - Z_1) + \left( \frac{1}{2} V_2^2 + 4fL + 2.225 \right) V_2^2
\]

Now inserting the numerical values where the elevation change was given as 100 ft (30.48 m), the pipe length is 250 ft (76.2 m), the diameter of the pipe is 2 in. (0.051 m), along with \( f \) equal to 0.0041, and \( V_2 = 311.3 \text{ cm/s} = 3.11 \text{ m/s} \), we then obtain

\[
W_{\text{device}} = 9.8 \frac{m}{s^2} \times 30.48 \text{ m} + \left( \frac{2.725 + 4 \times 0.0041 \times 76.2 \text{ m}}{2 \times 0.051 \text{ m}} \right) \frac{3.11^2 \text{ m}^2}{s^2} = 443.6 \frac{m^2}{s^2}
\]

If this result is then multiplied by kg kg\(^{-1}\) (see Equation 4.82), then \( W_{\text{device}} = 443.6 \text{ J/kg} \). Next, we multiply this result by the mass flow rate of the water, which is calculated next:

\[
m = \frac{\pi d^2}{4} \rho V_2 = \frac{\pi \times 0.051^2 \text{ m}^2}{4} \times 1000 \text{ kg/m}^3 \times 3.11 \frac{m}{s} \times 6.35 \frac{kg}{s} = 196
\]
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So, the total amount of work required to pump the water is given by

\[ W_{\text{total device}} = m \dot{W}_{\text{device}} = 6.35 \frac{\text{kg}}{\text{s}} \times 443.6 \frac{\text{J}}{\text{kg}} = 2816.6 \frac{\text{J}}{\text{s}} = 2816.6 \text{ W} \]

Because of the inefficiencies of the pump, this value needs to be divided by the efficiency of the pump (here 0.80) to find the actual work required. This would be the rating of the pump. So, dividing this result by the efficiency gives a value of \( W_{\text{actual}} = 3.52 \text{ kW} \), since 1 kW = 1000 W.

In terms of horsepower (HP), which is a common unit of power still used in the United States, we multiply this result by the conversion factor of 1.341 HP kW\(^{-1}\) and obtain \( W_{\text{actual}} = 4.72 \text{ HP} \).

Example 4.16

Consider the design of a power injector that rapidly injects a bolus of imaging contrast agent into a blood vessel. The diameter of the injector barrel is 2.5 cm and this is connected to a catheter with an inside diameter of 0.98 mm and a total length of 50 cm. Calculate the pressure (PSI) inside the power injector barrel and the force (N and lbf) required to deliver a flow rate of the contrast agent of 8 cm\(^3\) s\(^{-1}\) through the catheter. The contrast agent has a viscosity of 2.5 cP and a density of 1 g cm\(^{-3}\). The gauge pressure in the blood vessel is equal to 8 mmHg (gauge pressure\(^*\)). Assume the power injector is horizontal and at the same level as the injection site on the patient’s arm. Also, you can neglect any frictional force developed between the power injector’s plunger and the barrel wall that encloses the contrast agent within the power injector. In addition, the pressure losses due to fluid motion within the barrel itself are negligible in comparison to the pressure loss within the catheter and the pressure loss due to the contraction of the fluid as it enters the catheter. This means the pressure of the contrast agent fluid within the power injector barrel is constant.

Solution

We let location (1) be the contrast agent within the barrel of the power injector and location (2) is the plane at the catheter exit where the contrast agent leaves and enters the blood vessel. We also assume that the velocity of the contrast agent within the catheter, i.e., \( V_2 \), is much greater than the velocity of the contrast agent in the barrel of the power injector during the injection process, i.e., \( V_1 \). With these assumptions, the Bernoulli equation, as given by Equation 4.81, simplifies to give

\[ \frac{P_1}{\rho} = \frac{P_2}{\rho} + \frac{\alpha_2 V_2^2}{2} + h_{\text{friction}} \]

From Equation 4.84 we can write \( h_{\text{friction}} \) as

\[ h_{\text{friction}} = 4f \frac{LV_2^2}{2d} + \frac{V_2^2}{2} K_{\text{contraction}} \]

\(^*\) Recall that gauge pressure is that pressure relative to the local atmospheric pressure. Absolute pressure is gauge pressure plus local atmospheric pressure.
Using Equation 4.86 we can calculate the value of $K_{\text{contraction}}$. Since the diameter of the power injector barrel is much larger than the diameter of the catheter, we have that $K_{\text{contraction}} = 0.45$. The velocity of the contrast agent in the catheter tube ($V_2$) is

$$V_2 = 8 \frac{\text{cm}^3}{\text{s}} \times \frac{4}{\pi (0.098 \text{ cm})^2} = 1060.6 \frac{\text{cm}}{\text{s}}$$

With this value of $V_2$, we can calculate the Reynolds number as

$$Re = \frac{\rho d V_2}{\mu} = \frac{1 - \frac{\text{g}}{\text{cm}^3} \times 0.098 \text{ cm} \times 1060.6 \frac{\text{cm}}{\text{s}}}{2.5 \text{ cP} \times \frac{0.01 \text{ g}}{	ext{cm s cP}}} = 4157.5$$

The flow in the catheter is therefore turbulent since $Re > 2300$. Using Equation 4.85, we calculate the friction factor and obtain $f = 0.0099$. Next, we can calculate the value of $h_{\text{friction}}$:

$$h_{\text{friction}} = 4f \frac{1}{2d} \frac{V_2^2}{2} K_{\text{contraction}}$$

$$= \left[ 2 \times 0.0099 \times \frac{50 \text{ cm}}{0.098 \text{ cm}} + 0.5 (0.45) \right] \left( \frac{1060.6 \frac{\text{cm}}{\text{s}}}{s} \right)^2 = 1.162 \times 10^7 \frac{\text{cm}^2}{\text{s}^2}$$

The pressure in the vein ($P_2$) is given as 8 mmHg and this is equal to 1066.6 Pa. We can then use the Bernoulli equation to calculate the pressure within the power injector, i.e., $P_1$ as

$$P_1 = P_2 + \frac{\rho V_2^2}{2} + ph_{\text{friction}} = 1066.6 \text{ Pa} + \frac{1}{2} \frac{\text{g}}{\text{cm}^3} \times \left( \frac{1060.6 \frac{\text{cm}}{\text{s}}}{s} \right)^2 + 1 \frac{\text{g}}{\text{cm}^3} \times 1.162 \times 10^7 \frac{\text{cm}^2}{\text{s}^2}$$

$$= 1066.6 \text{ Pa} + \left( 5.624 \times 10^5 + 1.162 \times 10^7 \right) \frac{\text{g}}{\text{cm}^3} \times \frac{100 \text{ cm}}{\text{m}} \times \frac{1 \text{ kg}}{1000 \text{ g}} \times \frac{1 \text{ Pa}}{\frac{\text{kg}}{\text{ms}^2}}$$

$$P_1 = 1.22 \times 10^6 \text{ Pa} = 12.03 \text{ atm} = 176.9 \text{ PSI}$$

Now we can calculate the force needed to push the plunger

$$F = P_1 \times A_{\text{plunger}} = 1.22 \times 10^6 \text{ Pa} \times \frac{\frac{N}{\text{Pa}}}{\frac{m^2}{4}} \times \pi \left( \frac{2.5 \text{ cm}}{100 \text{ cm}} \right)^2 = 598.9 \text{ N} = 134.6 \text{ lb}$$

The next example illustrates a pseudo-steady-state application* of the Bernoulli equation to estimate the draining time of an IV bag.

* Problem 4.7 at the end of this chapter discusses an unsteady-state solution.
Example 4.17

The simplest patient infusion system is that of gravity flow from an intravenous (IV) bag. A 500 mL IV bag containing an aqueous solution is connected to a vein in the forearm of a patient. Venous pressure in the forearm is about 8 mmHg. The IV bag is placed on a stand such that the entrance to the tube leaving the IV bag is exactly 1 m above the vein into which the IV fluid enters. The length of the IV bag is 30 cm. The IV is fed through a tube with an internal diameter of 0.953 mm and the total length of the tube is 2 m. Calculate the flow rate of the IV fluid. Also estimate the time needed to empty the bag.

Solution

We apply the Bernoulli equation from the surface of the fluid in the IV bag (“1”) to the entrance to the vein (“2”). We expect the flow of the fluid through the bag and the tube to be laminar and therefore neglect the contraction at the entry to the feed tube and the expansion at the vein. The pressure at the surface of the fluid in the bag will be atmospheric ($P_1 = 760$ mmHg absolute), since the bag collapses as the fluid leaves the bag. The venous pressure, or $P_2$, is 8 mmHg gauge or 768 mmHg absolute. Because the fluid takes some length of time to leave the bag, we neglect the velocity of the surface of the fluid in the bag in comparison to the fluid velocity at the exit of the tube. Therefore, we assume $V_1 = 0$ and $V_2 \gg V_1$. We also set the reference elevation as the entrance to the patient’s arm; hence $Z_1 = 0$. Therefore, $Z_2$ is equal to the elevation of the bag relative to the position where the fluid enters the patient’s arm. This would equal 1 m plus the 30 cm length of the bag. Since there are no work devices in the system, $W_{device} = 0$. We can now write the Bernoulli equation for this particular problem as

$$\frac{P_1}{\rho} + gZ_1 = \frac{P_2}{\rho} + V_2^2 + \left(4\kappa \frac{L}{D}\right)\frac{V_2^2}{2}$$

Recall that the friction factor for laminar flow in a cylindrical tube is equal to $16/Re$. We can substitute this relationship into the above equation to obtain the following quadratic equation that can be solved for the exiting velocity, $V_2$:

$$V_2^2 + \left(\frac{32 \mu L}{\rho D^2}\right)V_2 - \left[gZ_1 + \frac{1}{\rho}(P_1 - P_2)\right] = 0$$

This equation may now be solved for the exit velocity recognizing that this quantity must be positive:

$$V_2 = \frac{-\left(\frac{32 \mu L}{\rho D^2}\right) + \left(\frac{32 \mu L}{\rho D^2}\right)^2 + 4 \left[gZ_1 + \frac{1}{\rho}(P_1 - P_2)\right]}{2}$$

$$= \frac{\left(\frac{32 \mu L}{\rho D^2}\right)^2 + 4 \left[gZ_1 + \frac{1}{\rho}(P_1 - P_2)\right]}{2}$$
Assuming the IV fluid has the same properties as water, and substituting the appropriate values for the parameters in this equation, the exit velocity is calculated as shown here. Note that $P_1 - P_2$ equals $-8$ mmHg which is equal to $-0.0105$ atm or $-1066.58$ kg m$^{-2}$:

$$V_2 = \left( \frac{32 \times 0.001 \text{ kg m s}^{-2} \times 2.0 \text{ m}}{1000 \text{ kg m}^{-3} \times 0.000953 \text{ m}^2} \right) + \left( \frac{32 \times 0.001 \text{ kg m s}^{-2} \times 2.0 \text{ m}}{1000 \text{ kg m}^{-3} \times 0.000953 \text{ m}^2} \right)^2 + 4 \left( \frac{9.8 \text{ m} \times 1.3 \text{ m} - \frac{1066.58 \text{ kg m}^{-2}}{1000 \text{ kg m}^{-3}}} {1000 \text{ kg m}^{-3}} \right)^{1/2}$$

$$V_2 = 0.1652 \frac{\text{ m}}{\text{ s}} \times \frac{100 \text{ cm}}{\text{ m}} \times \frac{60 \text{ s}}{\text{ min}} = 991.49 \frac{\text{ cm}}{\text{ min}}$$

$$Q = 991.49 \frac{\text{ cm}}{\text{ min}} \times \frac{\pi}{4} (0.0953 \text{ cm})^2 \times \frac{\text{ mL}}{\text{ cm}^3} = 7.072 \frac{\text{ mL}}{\text{ min}}$$

The time to empty the 500 mL bag based on this fluid flow rate of 7.072 mL min$^{-1}$ is then given by

$$t_{\text{empty}} \approx \frac{V_{\text{full}}}{Q} = \frac{500 \text{ mL}}{7.072 \frac{\text{ mL}}{\text{ min}}} = 70.7 \text{ min}$$

With the exit velocity of the fluid now estimated, we need to check the Reynolds number to see if our assumption of laminar flow is valid:

$$Re = \frac{\rho DV_2}{\mu} = \frac{1000 \text{ kg m}^{-3} \times 0.000953 \text{ m} \times 0.1652 \frac{\text{ m}}{\text{ s}}}{0.001 \frac{\text{ kg}}{\text{ ms}}} = 157$$

Since the $Re < 2300$, our assumption of laminar flow is correct.

### 4.11 Capillary rise and capillary action

Numerous processes depend on capillary action, i.e., the ability of liquids to penetrate freely into small pores, cracks, and openings. Capillary action is responsible for transporting water to the uppermost parts of tall trees and has a variety of applications in the fields of printing, textiles, agriculture, cleaning and sanitation products, and medical devices.

#### 4.11.1 Equilibrium capillary rise

Consider the situation shown in Figure 4.13. A small capillary tube is placed within a liquid. The liquid is drawn into the capillary as a result of the surface forces acting on the liquid wetting the inside surfaces of the capillary tube. These surface forces cause a curvature, called a meniscus, in the liquid surface as shown at position 3 in Figure 4.13, which, according to the Laplace-Young equation
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(i.e., Equation 2.185), lowers the pressure there relative to that outside of the capillary tube. This creates a suction that draws the fluid into the capillary tube. The liquid continues to rise up in the capillary until the forces tending to draw up the liquid are balanced by the downward force of gravity acting on the fluid.

The angle (θ) at the meniscus between the liquid surface and the wall of the capillary is called the contact angle. The radius of curvature of the meniscus (r) is related to the radius of the capillary tube (R) and the contact angle by the following equation:

\[ r = \frac{R}{\cos \theta} \]  

(4.88)

The equilibrium capillary rise (h) can be found from an analysis of the pressures at points 1, 2, 3, and 4, shown in Figure 4.13. From the Laplace-Young equation that we developed in Chapter 2 (i.e., Equation 2.185), we can write at the meniscus interface that

\[ P_1 - P_4 = \frac{2 \gamma}{r} \]  

(4.89)

The pressure at point 2 is greater than the pressure at point 3 by an amount equal to \( \rho_L gh \) and the pressure at point 1 is greater than the pressure at point 4 by \( \rho_L gh \); hence we can write that \( P_1 = P_4 + \rho_L gh \), \( P_2 = P_4 + \rho_L gh \). We also have the requirement that at equilibrium, \( P_1 = P_2 \). Therefore, \( P_4 - P_3 = (\rho_L - \rho_v)gh \). Using this result, Equation 4.89 can be solved for the capillary rise as shown in the following equation, recognizing that \( \rho_L \gg \rho_v \):

\[ h = \frac{2 \gamma \cos \theta}{\rho_L R g} \]  

(4.90)

Note that this equation also provides a simple means to determine the surface tension (\( \gamma \)) of a liquid by measuring its capillary rise.

Figure 4.13 Capillary rise of a liquid in a small diameter tube of radius R.
Example 4.18
Calculate the capillary rise for water in a tube with a diameter of 1 mm. Assume that \( \cos \theta \approx 1 \) and that the surface tension of water is 72 mN m\(^{-1}\).

Solution
Using Equation 4.90, we can calculate the capillary rise:

\[
h = \frac{2 \times 72 \times 10^{-3} \text{ Nm}^{-1} \times 1}{0.0005 \text{ m} \times 1000 \text{ kgm}^{-3} \times 9.8 \text{ msec}^{-2}} = 0.0294 \text{ m} = 29.4 \text{ mm} = 1.16 \text{ in.}
\]

4.11.2 Dynamics of capillary action
Suppose it is desired to estimate the rate at which the fluid enters the capillary, i.e., find how the capillary flow rate, \( Q \), and the rise height, \( h \), depend on time. For the situation shown in Figure 4.13, the liquid is drawn into the capillary by forces arising from the surface tension and this capillary force is retarded by the inertial force due to the mass of the rising fluid, the viscous force, and the gravitational force acting on the fluid. In a general sense, this is a very difficult problem; however, an approximate solution can be obtained if we assume that the flow in the capillary tube is laminar (Re < 2300) and that the velocity profile maintains the same parabolic shape (i.e., Equation 4.7) as the liquid is drawn into the capillary tube.

The capillary force draws fluid into the tube as a result of the suction pressure developed between positions 4 and 3 of Figure 4.13. This is given by the following expression, where we have also made use of Equations 4.88 and 4.89:

\[
F_{\text{capillary force}} = \pi R^2 (P_t - P_3) = \pi R^2 \frac{2y}{r} = 2\pi R \cos \theta \quad (4.91)
\]

The gravitational force acts on the mass of fluid (\( \rho_l \pi R^2 h(t) \)) within the tube at time \( t \). This is given by the following equation where \( \rho_l \) is the density of the fluid:

\[
F_{\text{gravitational force}} = \rho_l g \pi R^2 h(t) \quad (4.92)
\]

The viscous force arises as a result of the flow of the fluid as it is drawn into the capillary tube by the capillary force. It is assumed that the flow is laminar and that the Hagen–Poiseuille law (Equation 4.10) can be used to describe this unsteady flow. The driving force for this flow of fluid into the capillary tube is the pressure drop between points 2 and 3 as shown in Figure 4.13. Since the fluid is incompressible, the average velocity of the fluid (i.e., \( V \)) is the same as the observed meniscus velocity, which is \( dh(t)/dt \). With these assumptions, the volumetric flow rate of the fluid at time \( t \) is given by

\[
Q(t) = \pi R^2 V(t) = \pi R^2 \frac{dh(t)}{dt} = \frac{\pi R^4 (P_t - P_3)}{8 \mu h(t)} \quad (4.93)
\]
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The wall shear stress (i.e., \( \tau_w \)) is also related to \( (P_2 - P_3) \) through Equation 4.4, so we can solve this equation for \( (P_2 - P_3) \) in terms of the wall shear stress as shown here:

\[
(P_2 - P_3) = \frac{2h(t) \tau_w}{R}
\]  
(4.94)

Substituting Equation 4.94 into Equation 4.93 and solving for \( dh(t)/dt \) gives the next result:

\[
\frac{dh(t)}{dt} = \frac{2\pi R h(t) \tau_w}{8\pi \mu h(t)}
\]  
(4.95)

The viscous force is the wall shear stress times the circumferential area, which is the numerator of the right-hand side of Equation 4.95. Therefore, we obtain

\[
F_{\text{viscous force}} = 8\pi \mu h(t) \frac{dh(t)}{dt}
\]  
(4.96)

The inertial force is the mass of the fluid in the capillary tube multiplied by its acceleration. This is given by Newton’s second law:

\[
F_{\text{inertial force}} = \lim_{t \to 0} \left( mV_{t=0} - mV_{h(t)} \right) = \frac{d}{dt} \left( m \frac{dh(t)}{dt} \right) = \rho_l \pi R^2 \frac{d}{dt} \left( h(t) \frac{dh(t)}{dt} \right)
\]  
(4.97)

The inertial force is then equal to the sum of all the forces acting on the fluid as it rises in the tube through capillary action. The capillary force draws the fluid into the capillary tube and the viscous and gravitational forces work in opposition. Using the expressions developed earlier, we can then write that

\[
\rho_l \pi R^2 \frac{d}{dt} \left( h(t) \frac{dh(t)}{dt} \right) = 2\pi R \cos \theta - 8\pi \mu h(t) \frac{dh(t)}{dt} - \rho_l g \pi R^2 h(t)
\]  
(4.98)

Next, we rearrange Equation 4.98 and obtain what is known as the Bosanquet equation (Zhmud et al., 2000; Kornev and Neimark, 2001):

\[
\frac{d}{dt} \left( h(t) \frac{dh(t)}{dt} \right) + \frac{8 \mu}{\rho R^2} h(t) \frac{dh(t)}{dt} = 2 \frac{\gamma \cos \theta}{\rho_l R} - gh(t)
\]  
(4.99)

Equation 4.99 can then be solved for the rise of the fluid in the capillary tube as a function of time, provided suitable initial conditions can be defined. The initial conditions can be found by considering a solution to Equation 4.99 that is valid for short contact times when penetration of the fluid just begins. In this case, only the inertial and capillary forces are dominant and Equation 4.99 becomes

\[
\frac{d}{dt} \left( h(t) \frac{dh(t)}{dt} \right) = 2 \frac{\gamma \cos \theta}{\rho_l R}
\]  
(4.100)
Integration of Equation 4.100 with the initial condition that \( h(0) = 0 \) gives

\[
\frac{dh(t)}{dt} = \left( \frac{2 \gamma \cos \theta}{\rho_l R} \right) t \tag{4.101}
\]

The left-hand side of Equation 4.101 is proportional to the fluid momentum, \( \rho_l \pi R^2 h(t) \frac{dh(t)}{dt} \), and shows that as \( t \to 0 \), the momentum of the fluid approaches zero. Since \( h(0) = 0 \), this implies for \( t \to 0 \) that there is a finite velocity during the initial fluid entry phase which is known as the Bosanquet velocity \( (U_B) \) that is defined by the following relationships:

For \( t \to 0 \), \( \frac{dh(t)}{dt} \approx U_B \) or \( h(t) \approx U_B t \) \( \tag{4.102} \)

Equation 4.102 predicts that during the initial time of fluid penetration, the capillary rise increases linearly with time. Upon substitution of the results from the previous equation into Equation 4.101, we can solve for the initial fluid velocity as it enters the capillary tube due to the capillary force:

\[
U_B = \left( \frac{2 \gamma \cos \theta}{\rho_l R} \right)^{1/2} \tag{4.103}
\]

Hence, from Equations 4.102 and 4.103, the initial conditions for Equation 4.99 are \( h(0) = 0 \) and \( \frac{dh(0)}{dt} = U_B \).

At long times, the fluid in the capillary reaches a stationary level, which represents a balance between the capillary forces and the gravitational forces. At this equilibrium, \( \frac{dh(t)}{dt} = 0 \), and Equation 4.99 simplifies to the following equation for the capillary rise, which is the same as Equation 4.90 that was found earlier:

\[
h_{equilibrium} = \frac{2 \gamma \cos \theta}{\rho_l R g} \tag{4.104}
\]

For intermediate times, the first term representing the fluid inertia in Equation 4.99 can be neglected and we then obtain the Lucas–Washburn equation. This equation describes the rise of the fluid after the initial entry of the fluid into the tube, since the fluid acceleration is decreasing and the inertial force is much smaller for these times than the viscous and gravitational forces:

\[
\left( \frac{8\mu}{\rho_l R^2} \right) h(t) \frac{dh(t)}{dt} = \frac{2 \gamma \cos \theta}{\rho_l R} - gh(t) \tag{4.105}
\]

If the capillary rise is not large, then the gravitational force can also be neglected, and Equation 4.105 can be integrated to give the following result for the capillary rise as a function of time:

\[
h(t) = \sqrt{\frac{R \gamma \cos \theta t}{2 \mu}} \tag{4.106}
\]
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This equation predicts that the capillary rise is directly proportional to $\sqrt{t}$. Also, since $V(t) = \frac{dh(t)}{dt}$ and $Q(t) = \pi R^2 V(t)$, we obtain the following equations for the average velocity and the volumetric flow rate after differentiating Equation 4.106 with respect to $t$:

$$V(t) = \frac{\gamma R \cos \theta}{8 \mu} \quad \text{and} \quad Q(t) = \pi R^2 \frac{\gamma R \cos \theta}{8 \mu}$$  \hspace{1cm} (4.107)

Equation 4.107 predicts that for long times, $V$ and $Q$ decrease in proportion to $1/\sqrt{t}$. Notice also that Equation 4.107 at $t = 0$ gives the result that $V$ and $Q$ are infinite. This is a result of the neglect of the inertial forces when the fluid is first being drawn into the capillary tube.

Example 4.19

Estimate the time for water at 25°C to reach a height of 15 mm in a capillary tube that has a diameter of 1 mm. Assume that $\cos \theta \approx 1$ and that the surface tension of water is 72 mN m$^{-1}$.

Solution

Using Equation 4.106, we can solve for the time for the fluid to reach a particular height:

$$t = \frac{2h^2}{R \gamma \cos \theta} = \frac{2 \times 0.001 \text{ Nsm}^{-2} \times 0.015^2 \text{ m}^2}{0.0005 \text{ m} \times 72 \times 10^{-3} \text{ Nm}^{-1} \times 1} = 0.0125 \text{ s} = 12.5 \text{ ms}$$

Problems

4.1 Derive Equations 4.18 and 4.27.
4.2 Derive Equation 4.30.
4.3 Derive Equations 4.40 and 4.41.
4.4 Derive Equation 4.64 and then Equation 4.75.
4.5 Derive Equation 4.66.
4.6 Starting with Equations 4.51 and 4.52 work the steps to obtain Equation 4.59.
4.7 In Example 4.17, we obtained an estimate of the time needed to drain the IV bag. However, the IV bag has a length of 30 cm, and as the fluid drains from the bag, the potential energy ($Z_1$) of the remaining fluid that drives the flow will change with time. Therefore, to obtain a better estimate of the time to drain the bag, we also need to include how $Z_1$ changes with time. This can be obtained by combining the expression for the exit velocity ($V_2$) derived in Example 4.17 with an unsteady mass balance on the fluid in the bag itself. Letting $M(t)$ denote the mass of IV fluid remaining in the bag at any time $t$, we can write the IV bag mass balance as follows:

$$\frac{dM(t)}{dt} = \rho_L S_{\text{bag}} \frac{d(Z(t) - H)}{dt} = -\rho_L \frac{d_{\text{tube}}^2}{4} V_2$$

In this equation, $H$ represents the height of the catheter tube leaving the bottom of the bag relative to where this tube then enters the patient’s arm. For this problem, $H$ is equal to 1 m and $Z_1 - H$ is then the depth of the remaining fluid in the bag. $S_{\text{bag}}$ is the cross-sectional area.
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of the IV bag, and \( d_{\text{tube}} \) is the diameter of the tube. Use this equation, and the expression from Example 4.17 for \( V_2 \) to obtain the time to just drain the bag. How does this time compare to the pseudo-steady-state estimate of drain time obtained in Example 4.17?

4.8 Derive Equation 4.10. Start with Equation 4.18 and the assumption of a Newtonian fluid. Also show that Equation 4.10 can be obtained by integrating \( v_z(r) \) in Equation 4.7 using Equation 4.16.

4.9 The cardiac output in a human is about 6 L min\(^{-1}\). Blood enters the right side of the heart at a pressure of about 0 mmHg gauge and flows via the pulmonary arteries to the lungs at a mean pressure of 11 mmHg gauge. Blood returns to the left side of the heart through the pulmonary veins at a mean pressure of 8 mmHg gauge. The blood is then ejected from the heart through the aorta at a mean pressure of 90 mmHg gauge. Use the Bernoulli equation to obtain an estimate of the total work performed by the heart. Carefully state any assumptions and express your answer in watts.

4.10 Use the Bernoulli equation to describe the expected velocity and pressure changes upstream, within, and downstream of an arterial stenosis. A stenosis is a partial blockage or narrowing of an artery by formation of plaque (atherosclerosis).

4.11 Blood is flowing through a bundle of hollow fiber tubes that are each 50 \( \mu \text{m} \) in diameter. There are 10,000 tubes in the bundle. The hollow fiber tube length is 12 cm and the pressure drop across each tube is found to be 250 mmHg. The hematocrit of the blood is 0.40. Estimate the blood flow rate for these conditions in each tube.

4.12 Blood enters a hollow fiber unit that is used as an artificial kidney, i.e., for hemodialysis. The unit consists of 10,000 hollow fibers arranged in a shell and tube configuration. Blood flows from an artery in the patient’s arm through a catheter tube and is uniformly distributed to the fibers via an arterial head space region at the entrance of the unit. The blood then leaves each fiber through the venous head space region of the unit and is returned to a vein in the patient’s arm. Each hollow fiber has an inside diameter of 220 \( \mu \text{m} \) and a length of 25 cm. Assuming the maximum available pressure drop across the hollow fiber unit is 90 mmHg, estimate the total flow rate of blood through the hollow fiber unit.

4.13 You are designing a hollow fiber unit. The fiber diameter is 800 \( \mu \text{m} \) and their length is 30 cm. You want a flow rate of 8 mL min\(^{-1}\) for each fiber. What should be the pressure drop in mmHg across each fiber length to achieve this flow rate?

4.14 Using the data shown in Figure 4.5, find the best values of \( s \) and \( \tau_y \) in the Casson equation that fit these data. Recall that the shear stress can be found from the data shown in Figure 4.5 from the following relationship, \( \tau = \mu_{\text{apparent}} \dot{\gamma} \). Also, from the Casson equation, note that a plot of \( \tau^{1/2} \) versus \( \dot{\gamma}^{1/2} \) should be linear with a slope equal to \( s \) and an intercept of \( \tau_y^{1/2} \). Express the units of \( s \) as (dynes s cm\(^{-2}\))\(^{1/2}\) and \( \tau_y \) in (dynes cm\(^{-2}\)).

4.15 The following values were obtained for the apparent viscosity of blood (H = 40%) in tubes of various diameters. Estimate the thickness of the plasma layer (\( \delta \)) in microns and the core viscosity in cP from these data. The viscosity of the plasma is 1.09 cP.

<table>
<thead>
<tr>
<th>Tube Radius, R, ( \mu\text{m} )</th>
<th>Apparent Viscosity, cP</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.68</td>
</tr>
<tr>
<td>40</td>
<td>2.25</td>
</tr>
<tr>
<td>60</td>
<td>2.49</td>
</tr>
<tr>
<td>100</td>
<td>2.88</td>
</tr>
<tr>
<td>300</td>
<td>3.00</td>
</tr>
</tbody>
</table>
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4.16 A bioartificial liver has a plasma flow of 1000 mL min⁻¹ through a hollow fiber unit that contains hepatocytes on the shell side. The hollow fiber unit contains 10,000 fibers. The fiber length is 75 cm and the inside diameter of the fibers is 300 μm. What is the pressure drop across each fiber in mmHg?

4.17 A design for a novel aortic cannula for use on a blood pump oxygenator consists of a smooth thin-walled polyethylene tube 7 mm in diameter and 40 cm in length. For a flow rate of blood of 5 L min⁻¹ through the cannula, estimate the pressure drop (mmHg) over the length of the cannula.

4.18 For a cell-free plasma layer of 3 μm and a blood hematocrit of 40%, calculate the apparent viscosity for blood flowing in a 100 μm diameter tube. Assume the plasma viscosity is 1.093 cP and the core viscosity is 3.7 cP.

4.19 You are designing a small implantable microfluidic pump for the continuous delivery of a drug. The pump is a two compartment cylindrical chamber; one compartment containing the drug dissolved within a solvent, and the other compartment is the pump engine. These two compartments are separated by a movable piston that pushes on the drug compartment as the pump engine operates. At the other end of the drug compartment, there is an exit tube through which the drug solution flows. The exit tube of the pump through which the drug leaves the pump has an internal diameter of 10 μm and a total length of 15 cm. What gauge pressure is needed in mmHg within the drug compartment to maintain a flow rate of the drug solution of 350 μg day⁻¹? You may assume that the drug solution has a density of 1 g cm⁻³ and a viscosity of 3 cP.

4.20 You are part of a team developing an osmotic pump for the delivery of a drug. An osmotic pump has two compartments; one compartment, the osmotic engine, contains an osmotic agent that is retained by a membrane and imbibes water when placed within the body. This compartment also has a piston that expands against another compartment containing the drug solution as water is imbibed from the surroundings. The movement of the piston then forces the drug solution out into the body. A question has been raised as to what would be the maximum pressure within the device if after implantation the exit tube that delivers the drug becomes blocked? Assume the interstitial fluid pressure is –3 mmHg and its osmotic pressure is 8 mmHg. If the concentration of the osmotic agent is 0.05 OsM, what would be the maximum hydrodynamic pressure within the device assuming the delivery tube becomes blocked?

4.21 A viscometer has been used to measure the viscosity of a fluid at 20°C. The data of the shear stress versus the shear rate when plotted on a log-log graph is linear. Is this fluid Newtonian? Explain your answer.

4.22 Blood flows through a bundle of hollow fibers at a total flow rate of 250 mL min⁻¹. There are a total of 7500 fibers. The diameter of each fiber is 75 μm and the length of a fiber is 15 cm. What is the pressure drop across a fiber in mmHg?

4.23 Commercially available spermicidal or contraceptive gels have been developed for the purpose of preventing sperm transport and thus blocking fertilization (Owen et al., 2000). Current interest has also lead to the possible use of contraceptive gels as a means to reduce the spread of sexually transmitted diseases such as AIDS, and has led to interest in developing formulations of these gels for both prophylaxis and contraception. The physical properties of these gels must be such that, when applied, they spread to coat the vaginal epithelia, and then stay in place long enough to provide contraception, as well as adequate protection from disease causing agents such as bacteria and viruses. This is accomplished by gel formulations that
deliver topically bioactive compounds, such as microbiocides, and also by the physical barrier to infection provided by the coating layer. The spreading and retention of intravaginal contraceptive formulations are fundamental to their efficacy and these performance characteristics are governed in part by their rheological properties.

In vivo, these contraceptive gels will experience a wide range of shear rates as a result of movements of the vaginal epithelial surfaces, gravity, capillary flow, and sex. It is estimated that these shear rates may range from as low as 0.1/s to as high as 100/s during sex. The polymeric nature of these gels suggests that they will exhibit non-Newtonian rheological behavior.

A popular model for describing the rheological behavior of non-Newtonian gels is that of the two-parameter power law model. This model is shown by the following equations for flow in a cylindrical tube:

\[ \tau = m\gamma^n = -m\gamma^{n-1} \frac{dv}{dr} = -\mu_{\text{apparent}} \frac{dv}{dr} = \mu_{\text{apparent}} \dot{\gamma} \]

\[ \mu_{\text{apparent}} = m\gamma^{n-1} \]

In this equation, \( m \) and \( n \) are constants characterizing the fluid, and \( \dot{\gamma} \) is the shear rate equal to \( -\frac{dv}{dr} \). Using this equation, derive an expression for the axial velocity profile, \( v_z(r) \), and the mass flow rate \( \dot{m} = \rho \dot{V} \) of a fluid described by the power law model.

The following table shows data obtained from a rheometer for the commercially available gel called Conceptrol.

**Apparent Viscosity of Conceptrol versus Shear Rate** (Owen et al., 2000)

<table>
<thead>
<tr>
<th>Shear Rate, s(^{-1})</th>
<th>Viscosity, Pa s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>6000</td>
</tr>
<tr>
<td>0.05</td>
<td>2000</td>
</tr>
<tr>
<td>0.1</td>
<td>800</td>
</tr>
<tr>
<td>0.5</td>
<td>400</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>80</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>50</td>
<td>15</td>
</tr>
<tr>
<td>100</td>
<td>5</td>
</tr>
<tr>
<td>500</td>
<td>0.80</td>
</tr>
<tr>
<td>1000</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Perform a regression Analysis of the data in this table and find the power law parameters \( m \) and \( n \). Compare your model predictions to the data shown in this table. Carefully state the units these parameters have.

Bush, Petros, and Barrett-Lennard (J. Biomech., 30, 967–969, 1997) studied the flow of urine in an anatomical model of the human female urethra as shown in Figure 4.14.
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They obtained the following data shown in the table from their experimental model.

<table>
<thead>
<tr>
<th>Flow Rate, cm$^3$ s$^{-1}$</th>
<th>Pressure Difference, $\Delta P$, cm of H$_2$O</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.0</td>
</tr>
<tr>
<td>8</td>
<td>4.5</td>
</tr>
<tr>
<td>10</td>
<td>8.0</td>
</tr>
<tr>
<td>13</td>
<td>14.0</td>
</tr>
<tr>
<td>15</td>
<td>21.0</td>
</tr>
<tr>
<td>17</td>
<td>28.0</td>
</tr>
<tr>
<td>20</td>
<td>38.0</td>
</tr>
<tr>
<td>23</td>
<td>52.0</td>
</tr>
<tr>
<td>25</td>
<td>60.0</td>
</tr>
</tbody>
</table>

The pressure difference represents the reservoir head (H) shown in Figure 4.14 plus the additional 2 cm from the bottom of the reservoir to the exit of the urethra. Use the Bernoulli equation (Equation 4.81) to predict the pressure difference for each of the flow rates given in this table. Show your results as a graphical comparison between the data and the model. Assume that the diameter of the urethra is 3.25 mm and that its length is 4 cm. Determine whether the flow is laminar or turbulent. If the model does not fit the data, what parameters in the model can you change to improve the fit, e.g., what is the effect of the average urethral diameter?

4.25 Consider the steady laminar flow of a Newtonian fluid in the thin channel formed between two large parallel and horizontal plates of length L in the flow direction and width W. The plates

Figure 4.14 An in vitro fluid flow model of the human urethra. (From Bush, M.B. et al., *J. Biomech.*, 30, 967, 1997.)
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are separated by a distance of $2H$ and $H \ll L$ and $W$. Show that the velocity profile, $v_z(y)$, and the volumetric flow rate, $Q$, are given by the following expressions:

$$v_z(y) = \frac{(P_0 - P_L)H^2}{2 \mu L} \left[ 1 - \left( \frac{y}{H} \right)^2 \right]$$

$$Q = \frac{2}{3} WH^3 \frac{(P_0 - P_L)}{\mu L}$$

with $P_0$ and $P_L$ are the inlet and exit pressures.

4.26 Using the results from Problem 4.25, show that the following expression approximates the penetration of liquid, $L(t)$, by capillary action into a slit channel used in a diagnostic device:

$$L(t) = 2 \left[ \frac{H \gamma \cos \theta}{3 \mu} \right]^{1/2} t^{1/2}$$

A diagnostic device makes use of a thin rectangular channel to draw in a sample of blood. Assuming the blood sample has a viscosity of $3 \text{ cP}$ and that the plates forming the channel are separated by a distance of $1 \text{ mm}$, estimate the time for the sample of blood to travel a distance of $15 \text{ mm}$ in the channel. Assume the blood has a surface tension of $0.06 \text{ N m}^{-1}$ and that the contact angle is $70^\circ$.

4.27 The following table shows data for the measured velocity profile for the laminar boundary layer flow of a fluid across a flat plate (from Schlichting, 1979). Make a plot of these data and compare to the approximate velocity profile given by Equation 4.74.

<table>
<thead>
<tr>
<th>$y$ (mm)</th>
<th>$V$ (m/s)</th>
<th>$v_z(x,y)$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>0.175</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.34</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0.495</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.63</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>0.745</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>0.92</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>0.955</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>0.98</td>
<td></td>
</tr>
</tbody>
</table>

4.28 Zhmud et al. (2000) obtained the following data for the capillary rise of dodecane in a $200 \mu m$ diameter capillary tube. Compare these results to those predicted by the Lucas–Washburn
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equation, i.e., Equation 4.106. The physical properties for dodecane are as follows: viscosity = 1.7 × 10⁻³ Pa s, surface tension = 2.5 × 10⁻² N m⁻¹, and density = 750 kg m⁻³. What is the value of the contact angle that gives the best fit to these data using the Lucas–Washburn equation?

<table>
<thead>
<tr>
<th>Time, s</th>
<th>h(t), mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>2</td>
</tr>
<tr>
<td>0.06</td>
<td>4.3</td>
</tr>
<tr>
<td>0.1</td>
<td>6.1</td>
</tr>
<tr>
<td>0.13</td>
<td>7.9</td>
</tr>
<tr>
<td>0.16</td>
<td>9</td>
</tr>
<tr>
<td>0.20</td>
<td>10</td>
</tr>
<tr>
<td>0.23</td>
<td>11</td>
</tr>
<tr>
<td>0.26</td>
<td>12</td>
</tr>
<tr>
<td>0.30</td>
<td>12.8</td>
</tr>
<tr>
<td>0.33</td>
<td>13.5</td>
</tr>
<tr>
<td>0.36</td>
<td>14.2</td>
</tr>
<tr>
<td>0.40</td>
<td>14.8</td>
</tr>
<tr>
<td>0.42</td>
<td>15.5</td>
</tr>
<tr>
<td>0.46</td>
<td>16.1</td>
</tr>
<tr>
<td>0.50</td>
<td>16.3</td>
</tr>
</tbody>
</table>

4.29 In the paper by Zhmud et al. (2000), the capillary rise for diethyl ether in a 1 mm diameter capillary tube was found to be 8.6 mm at equilibrium. How does this value compare with the value predicted by Equation 4.90? The physical properties for diethyl ether are as follows: surface tension = 1.67 × 10⁻² N m⁻¹, density = 710 kg m⁻³, contact angle = 26°.

4.30 Prove that for laminar flow in a cylindrical tube that the kinetic energy correction factor, α, in the Bernoulli equation is equal to 2.

4.31 Show that \( f = \frac{16}{Re} \) for laminar flow in a cylindrical tube.

4.32 A small airplane is flying at 3000 m above sea level. The density of air at this altitude is 0.83 kg m⁻³. A Pitot tube gives a difference between the stagnation pressure and the static pressure of 50 mmHg. Based on this information, how fast is the airplane traveling in miles per hour?

4.33 The viscosity of a fluid (\( \mu_{\text{test}} \)) may be found in terms of the viscosity of another reference fluid whose viscosity is known (\( \mu_{\text{reference}} \)) by measuring the time it takes for the fluid (\( t_{\text{test}} \)) to drain by gravity a certain distance within a vertical tube of constant cross section, and then comparing that time to the time (\( t_{\text{reference}} \)) for the reference fluid whose viscosity is known. Show that for either fluid the velocity of the fluid exiting the tube is related to the change in height (\( z \)) of the fluid by the following equation:

\[
\frac{dz}{dt} = -V_2
\]

where \( V_2 \) is the velocity of the fluid exiting the bottom of the tube. Next, apply the Bernoulli equation to the fluid in the tube from the top of the fluid surface (1) to the bottom of the tube (2).
Assuming laminar flow of the fluid with $f = 16/Re$ and using the above equation for $V_2$, show that $dz/dt$ is also given by the following equation:

$$\frac{dz}{dt} = -\frac{\rho d^2 g}{32\mu}$$

where
- $d$ is the inside diameter of the tube
- $g$ is the acceleration of gravity

Next, integrate this equation and show that the time for the fluid to drain a distance equal to $\Delta z$ is given by

$$\Delta z = \frac{\rho D^2 g t}{32\mu}$$

Now, if both fluids drain the same distance, i.e., $\Delta z_{\text{test}} = \Delta z_{\text{reference}}$, show that the following equation may be used to relate their viscosities in terms of their drain times and densities:

$$\frac{\mu_{\text{test}}}{\mu_{\text{reference}}} = \frac{\rho_{\text{test}} t_{\text{test}}}{\rho_{\text{reference}} t_{\text{reference}}}$$

Below is shown some data for the drain time of different concentrations of chitosan in water that was obtained by one of my former PhD students, Prasanjit Das. Pure water was used as the reference fluid with a viscosity of 0.001 Pa s or 1 cP and a density of 997 kg m$^{-3}$. The time ($t_{\text{reference}}$) for the water to flow by gravity a defined distance ($\Delta z_{\text{test}} = \Delta z_{\text{reference}}$) in a capillary tube was found to be 0.120 s. Calculate the viscosity (cP) of the chitosan solution for each of the concentration values given in the following table.

<table>
<thead>
<tr>
<th>Chitosan Concentration, ppm</th>
<th>Chitosan Solution Density, kg m$^{-3}$</th>
<th>Drain Time ($t_{\text{test}}$), s</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>997</td>
<td>0.120</td>
</tr>
<tr>
<td>100</td>
<td>997</td>
<td>0.120</td>
</tr>
<tr>
<td>200</td>
<td>997</td>
<td>0.230</td>
</tr>
<tr>
<td>500</td>
<td>997</td>
<td>0.290</td>
</tr>
<tr>
<td>1000</td>
<td>998</td>
<td>0.480</td>
</tr>
</tbody>
</table>

4.34 The formation of a small droplet at the tip of a capillary tube can be used to determine the surface tension of a fluid. This is known as the hanging droplet method. The droplet can grow in size until the gravitational force exceeds the surface tension force that holds the droplet to the periphery of the tube, i.e.,

$$\gamma \pi D_{\text{tube}} = \frac{4}{3} \pi R^3 \rho_l g$$

where
- $\gamma$ is the surface tension of the fluid
- $R$ is the radius of the spherical droplet formed
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Show that the surface tension is then given by the following equation:

\[ \gamma = \frac{0.1667d^3 \rho_{lg}}{D_{tube}} \]

where \( d \) is the diameter of the droplet that just releases itself from the tube. Below is shown some data that was obtained by my former PhD student, Prasanjit Das, on droplets formed at the tip of a 1 mm outside diameter capillary tube for different concentrations of chitosan in water. From these data, calculate the surface tension (mN m\(^{-1}\)) of the chitosan solution for each of the concentration values given in the following table.

<table>
<thead>
<tr>
<th>Chitosan Concentration, ppm</th>
<th>Chitosan Solution Density, kg m(^{-3})</th>
<th>Droplet Diameter, mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>997</td>
<td>3.48</td>
</tr>
<tr>
<td>100</td>
<td>997</td>
<td>3.46</td>
</tr>
<tr>
<td>200</td>
<td>997</td>
<td>3.43</td>
</tr>
<tr>
<td>500</td>
<td>997</td>
<td>3.40</td>
</tr>
<tr>
<td>1000</td>
<td>998</td>
<td>3.33</td>
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</table>

4.35 Bazilevsky et al. (2003) measured the entry capillary flow rate of water into a 0.65 mm diameter capillary tube. The average entry flow rate of the water was found to be 220 mm\(^3\) s\(^{-1}\). What is the entrance velocity of the water and how does this compare to that predicted by the Bosanquet equation? Assume the surface tension of water is 0.071 N m\(^{-1}\) and its viscosity is 0.001 Pa s. Also \( \theta = 0^\circ \).

4.36 A concentrated solution of sugar dissolved in water is flowing through a capillary tube with an inside diameter of 2 mm. The capillary tube will be used to find the viscosity of this solution. The length of the capillary tube is 10 cm. The density of the solution is 1200 kg m\(^{-3}\). For a flow rate of 60 cm\(^3\) min\(^{-1}\), the pressure drop per length of the tube was found to be 1.0 mmHg cm\(^{-1}\). What is the viscosity (Pa s) of the fluid?

4.37 A polymeric fluid having a viscosity of 0.40 Pa s, a density of 800 kg m\(^{-3}\), and a surface tension of 0.02 N m\(^{-1}\) is drawn by capillary action into a glass tube of radius 0.025 cm. If the contact angle is such that the \( \cos \theta \sim 1 \), estimate the time required for the fluid to reach a height equal to 90% of its equilibrium rise. Also, find the initial flow rate of this fluid into the tube in cm\(^3\) s\(^{-1}\).

4.38 The capillary rise, \( h(t) \), as a function of time, \( t \), for a biofluid was measured in a capillary tube of radius 0.3 mm. When the data was plotted as \( h(t) \) versus \( t^{1/2} \) the data was found to be linear with a slope equal to 0.06 m s\(^{-1/2}\). If the fluid has a viscosity of 1.2 \times 10^{-3} \text{ Pa s}, and the contact angle ~0°, estimate the surface tension of the fluid in N m\(^{-1}\).

4.39 A particular fluid has a shear stress of 0.005 N m\(^{-2}\) at a shear rate of 1 s\(^{-1}\) and a shear stress of 2 N m\(^{-2}\) at a shear rate of 50 s\(^{-1}\). Present an argument as to whether this fluid is Newtonian or non-Newtonian.

4.40 Blood is flowing at a flow rate of 7 L min\(^{-1}\) through a tube that is 6 mm in diameter and 50 cm in length. Estimate the pressure drop of the blood over this length of tubing in mmHg.

4.41 The apparent viscosity of blood flowing in a 100 \( \mu \text{m} \) diameter tube was found to be 2.6 cP. Assuming that the core viscosity of blood is about 3.7 cP, estimate the thickness of the marginal zone layer. Assume the plasma viscosity is 1.1 cP.
4.42 Blood flows through a 100 μm diameter glass tube that is 0.1 cm in length. Estimate the volumetric flow rate of the blood in cm³ h⁻¹ if the pressure difference over the length of the tube is 6000 Pa.

4.43 You are designing a hollow fiber unit with 10,000 fibers. The fiber diameter is 1000 μm and their length is 50 cm. You want a flow rate of 100 L min⁻¹ for the entire unit. What should the pressure drop (mmHg) be across the unit to achieve this flow rate?

4.44 The broth from a continuous fermentor has a density of 1.02 g cm⁻³ and a viscosity of 1.8 cP. The broth is being pumped from the fermentor into the bottom of a filtration feed tank that leads into a filtration system. The difference in fluid levels within the fermentor and the filtration feed tank is 20 ft and remains constant since the filtration feed tank is continuously feeding a downstream filtration system at the same flow rate as the broth that is being pumped from the fermentor. The pressure in the fermentor is maintained at 1 atm and the pressure in the feed tank is maintained at 6 atm in order to facilitate the downstream filtration process. The pipe connecting the fermentor and the feed tank is equivalent to 75 ft of pipe with an inside diameter of 3 in., where, this equivalent length includes the additional resistance of any valves and pipe fittings. The desired flow rate of the fermentation broth as it is pumped to the feed tank is 100 gal min⁻¹. What power (kilowatts) must be delivered by the pump to the fluid in order to effect this transfer of broth from the fermentor to the feed tank?

4.45 Water at 20°C is pumped through 3000 cm of pipe with an internal diameter of 7.8 cm into an overhead storage tank that vents to the surroundings. The total elevation change is 1000 cm. The valves and other pipe fittings are equivalent to an additional pipe length equal to 15 pipe diameters. What outlet pressure in atmospheres of the pump is needed to move the water at a flow rate of 70 L min⁻¹? At 20°C the viscosity of water is 1.002 cP and the density is 0.9982 g cm⁻³.

4.46 The pressure drop of a nutrient fluid flowing through the hollow fibers of a bioreactor cannot exceed 50 mmHg. There are 10,000 fibers in the bioreactor, and the total flow rate of the nutrient media to the bioreactor is 50 L min⁻¹. Estimate the allowable length of these fibers assuming the hollow fiber radius is 0.1 cm and the viscosity of the nutrient fluid is 0.05 Pa s.

4.47 You are studying in the laboratory the laminar boundary layer flow of a fluid across a very thin flat plate. The flat plate is suspended vertically from a spring in the range where Hooke’s law applies, i.e., F = k Δλ, where Δλ is the amount of spring extension from the unloaded position of the spring and k is the spring constant. Under conditions of no air flow, the extension of the spring due to the weight of the flat plate is Δλ_plate = 5 cm. When there is an upward flow of air, the resulting drag opposes the weight of the flat plate. Find the air flow in m s⁻¹ for which the spring extension is zero, i.e., Δλ = 0. Use the following data to find your answer:
  - Mass of the plate, M = 0.001 kg
  - Plate length, L = 0.2 m
  - Plate width, W = 0.1 m
  - Air speed, V = find this
  - Air density, ρ = 1.2 kg m⁻³
  - Air kinematic viscosity, ν = 1.51 × 10⁻⁵ m² s⁻¹

4.48 Estimate the maximum power that is generated in megawatts from a wind turbine whose blades are 300 ft in diameter. Assume the wind speed averages 20 miles h⁻¹ and that the density of air is 1.2 kg m⁻³. Carefully state your assumptions.
4.49 Estimate the pressure drop in mmHg of blood flowing in the annulus formed by two concentric cylindrical tubes. The inner tube has an outer diameter of 8 mm and the outer tube has an inner diameter of 15 mm. The blood flow rate is 15,000 mL min⁻¹, and the overall length of the tube section being considered is 40 cm.

4.50 For the flow of a fluid through a small tube that is 0.9 mm in diameter, the flow rate was 0.3 cm³ s⁻¹ and the pressure drop over the length of the 50 cm tube was found to be 50 kPa. Estimate the viscosity of this fluid in Pa s.

4.51 Below is shown viscometry data obtained for a 5% aqueous solution of hydroxyethylcellulose (HEC). Choose the best answer that describes the rheological behavior of this solution:
   a. Newtonian
   b. Non-Newtonian
   c. Non-Newtonian power law solution (i.e., \( \mu = m \dot{\gamma}^{n-1} \))
   d. None of these choices

4.52 The equilibrium rise height of a rather viscous fluid in a small capillary tube of diameter equal to 0.1 mm was found to be 65 mm. The contact angle was also measured to be 75°. Estimate the surface tension of this fluid if its viscosity is \( \mu = 0.75 \) Pa s and \( \rho_L = 800 \) kg m⁻³.

4.53 Water at 70°F enters a pump through a 3 in. schedule 40 pipe at atmospheric pressure and is being pumped at a rate of 100 gal min⁻¹ (1 gal = 3.7853 L) through a pipe system made up of 500 ft of 3 in. schedule 40 steel pipe (internal diameter = 3.068 in.). The pipe circuit includes 4°–90° elbows. The total change in elevation from the pump entrance to the discharge of the water into the atmosphere is equal to 400 ft. Find the horsepower (HP) required for the pump assuming the pump efficiency is 70%. The viscosity of water under these conditions is 0.96 cP and the density of water is 1 g cm⁻³.

4.54 A Newtonian fluid with a density of 1.06 g cm⁻³ is flowing through a horizontal tube with a length of 50 cm and a diameter of 6 mm at a flow rate of 7.5 L min⁻¹. The pressure drop across this length of tube was found to be 200 mmHg. Estimate the viscosity of this fluid in cP.

4.55 A fluid with a density of 1.03 g cm⁻³ is flowing at the rate of 1.8 L h⁻¹ through a small tube that is 1.5 mm in diameter and 50 cm in length. If the viscosity of this fluid is 2.8 cP, estimate the pressure drop over the tube length in mmHg.

4.56 A Newtonian fluid with a density of 1.03 g cm⁻³ is flowing through a horizontal tube with a diameter of 4 mm at a flow rate of 15 L min⁻¹. The viscosity of this fluid is 1.1 cP. What pressure drop in mmHg would you expect over a 50 cm length of this tube?

4.57 Consider the rise of water in a capillary tube of radius equal to 50 μm. What is the equilibrium rise height in cm of the water after one end of the tube is immersed in water? Assume the viscosity of water is 0.001 Pa s, the surface tension of the water is 7.3 × 10⁻² N m⁻¹, and the density of the water is 1000 kg m⁻³. The surface of the tube is highly wettable so the contact angle is zero.

4.58 A woman decides to build a log cabin at the base of a tall mountain and is looking at the feasibility of tapping into a snow melt fed lake that is about 500 ft above her cabin for her water supply. The plan is to install a small pipeline that runs from the lake into a water storage tank that will then provide water as needed to her cabin which is located nearby. The change in elevation from the surface of the lake to where the water would empty into the storage tank is 475 ft. The plan is to use a 1.25 in. inside diameter pipe that would have an overall length of 600 ft. Included in the piping circuit is a priming pump (which only runs to get the water flowing, sort of like a siphon) with a \( K_{\text{pump}} = 1.5 \) and there are also 8°–90° elbows (\( K_{\text{elbows}} = 1.0 \)). When the water is flowing, what is the flow rate of the water in gallons per minute as it exits the
pipe into the top of the storage tank? Assume that the water exiting the pipe is not submerged in the water. Also assume that the tank is vented to the atmosphere. You can also neglect the difference in barometric pressure between the lake and the cabin. The water is at 41°F and its viscosity and density at this temperature are, respectively, 1.519 cP and 1000 kg m\(^{-3}\).

**4.59** A Newtonian fluid with a density of 1.10 g cm\(^{-3}\) and a viscosity of 1.5 cP is flowing through a horizontal tube with a diameter of 5 mm. If the pressure drop across the length of the tube cannot exceed 2000 mmHg, estimate the maximum length of the tube in cm. The flow rate of the fluid is 7 L min\(^{-1}\).

**4.60** A fluid with a density of 1.03 g cm\(^{-3}\) is flowing through a small horizontal tube that is 1.25 mm in diameter and 75 cm in length. If the viscosity of this fluid is 1.8 cP, and the pressure drop over this length of tube is 60 mmHg, estimate the flow rate of the fluid in mL h\(^{-1}\).

**4.61** A droplet of a nerve agent lands on the surface of a soldier’s clothing and covers an area of 2.8 cm\(^2\). The surface tension of the nerve agent is 0.028 N m\(^{-1}\). The material in the person’s clothing is such that the weave makes tiny cylindrical openings that have a diameter of 0.5 mm. The cylindrical openings cover 45% of the clothing area. The contact angle (\(\theta\)) of the agent and clothing material is such that the \(\cos \theta = 0.9063\). From this information, estimate the initial mass flow rate of the nerve agent into the clothing material in grams per second. The density of the nerve agent is 1.04 g cm\(^{-3}\).

**4.62** Blood flows through a vein that is 0.71 cm in diameter and 20 cm in length. If the blood has a viscosity of 4 cP and a density of 1.04 g cm\(^{-3}\), what is the flow rate of the blood in cm\(^3\) s\(^{-1}\) if the pressure drop over the length of the vein is equal to 1 mmHg?

**4.63** The rheology of an infant gruel formula is being evaluated in a capillary viscometer. The viscometer is a glass tube that has an internal diameter of 1.25 mm and is 5 cm in length. When the flow rate of the gruel was 1 cm\(^3\) min\(^{-1}\) the pressure drop over the length of the viscometer was found to be 137 mmHg. What is the apparent viscosity of the gruel in units of Pa s?

**4.64** A Newtonian fluid with a density of 1.06 g cm\(^{-3}\) is flowing through a horizontal tube with a length of 50 cm and a diameter of 6 mm at a flow rate of 7.5 L min\(^{-1}\). The viscosity of the fluid is equal to 2.55 cP. What is the pressure drop across this length of tube?

**4.65** A fluid with a density of 1.03 g cm\(^{-3}\) is flowing through a small tube that is 1.5 mm in diameter and 50 cm in length. If the viscosity of this fluid is 2.8 cP, and the pressure drop over the tube length is 42.3 mmHg, what is the flow rate of the fluid in L h\(^{-1}\)?

**4.66** Calculate the pressure drop in mmHg for the flow of a liquid material in a horizontal, smooth, and circular tube 1000 cm in length that has an inside diameter of 3 cm. The liquid material flows through the tube at a mass rate of 1028 g s\(^{-1}\) and has a density of 0.935 g cm\(^{-3}\) and a viscosity of 1.95 cP.

**4.67** A container of a viscous liquid makes contact with an open capillary tube that is vertical. The capillary tube has a diameter of 0.1 mm. The equilibrium rise height of this fluid in the capillary was found to be 0.0765 m. How long will it take for the viscous liquid to rise within the capillary tube to a height of 0.069 m? The contact angle of the liquid with the capillary tube is 60°, the viscosity of the liquid is 1 Pa s, and its density is 0.8 g cm\(^{-3}\).

**4.68** The equilibrium rise height of a rather viscous fluid in a small capillary tube of diameter equal to 0.1 mm was found to be 65 mm. The contact angle was also measured to be 75°. Estimate the surface tension of this fluid if its viscosity is \(\mu = 0.75\) Pa s and \(\rho_L = 800\) kg m\(^{-3}\).

**4.69** A rapid blood bag infusion system is being designed that involves effectively squeezing the bag of blood through a tube (1.5 mm internal diameter with a total length of 2 m) into a vein in the patient’s arm. The blood bag is placed within another pressurizing bag that is pressurized
The physical and flow properties of blood and other fluids with air that squeezes the blood bag at constant pressure. It is desired to empty the bag of blood using this method in 30 seconds. The bag contains 500 mL of blood. What pressure (mmHg) is needed in the pressurizing bag to make this possible? Assume the pressure in the patient’s vein is 8 mmHg and that the bag of blood is 0.5 m above the point of entry of the blood into the patient’s arm. The density of blood is 1.06 g cm\(^{-3}\) and its viscosity is 3 cP. Also neglect any frictional losses due to the flow of blood within the blood bag.

4.70 A Newtonian fluid with a viscosity of 1300 cP needs to be pumped through a tube that has an internal diameter of 1.25 mm and a length of 10 cm. If the flow rate of this fluid is 1 cm\(^3\) min\(^{-1}\), what is the pressure drop (in mmHg) over the length of the tube that is needed to produce this flow?

4.71 Estimate the radius of a capillary tube (mm) from these measurements taken for the flow of a viscous fluid through the tube: capillary tube length = 50.02 cm, kinematic viscosity = 4.03 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}, fluid density (\rho) = 0.9552 \times 10^3 \text{ kg m}^{-3}, pressure drop across the tube in a horizontal position = 4.766 atm, mass flow rate through the tube of the viscous fluid = 0.1798 kg min\(^{-1}\).

4.72 Calculate the pressure drop in mmHg for a fluid flowing within a smooth tube that is 0.8 cm in diameter and 20 cm in length. The fluid velocity in the tube is 200 cm s\(^{-1}\) and the fluid has a density of 1.04 g cm\(^{-3}\) and a kinematic viscosity of 0.04 cm\(^2\) s\(^{-1}\).

4.73 A small bubble of air in water (\gamma = 0.073 \text{ N m}^{-1}) has a radius of 0.10 mm. Find the difference in pressure in mmHg between the inside and outside of the bubble.

4.74 A rectangular conduit 200 cm in length is needed to convey water at a flow rate of 18 mL s\(^{-1}\). The conduit cross section has a height of 0.23 cm and a width of 0.45 cm. Calculate the pressure drop (mmHg) over the length of the conduit to obtain this flow rate. Assume the kinematic viscosity (\nu) of water is 0.01 cm\(^2\) s\(^{-1}\) and its density is 1 g cm\(^{-3}\).

4.75 Water at 16°C (viscosity 1.13 cP and density 1 g cm\(^{-3}\)) is pumped from a large lake to the top of a mountain through a 6 in. steel pipe at an average velocity of 10 ft s\(^{-1}\). The inside diameter of this schedule of pipe is 5.501 in. The pipe discharges into the atmosphere at a level 4000 ft above the level in the lake. The pipeline is 5000 ft in total length. You can neglect the pressure loss due to any valves and fittings. The overall efficiency of the pump and its motor is 70%. If electricity costs $0.11 per kWh, what will it cost to pump this water for 1 h?

4.76 A commercial size hollow fiber bioreactor contains 1000 tubes and each fiber has a length of 100 cm and an inside diameter of 6 mm. A Newtonian culture fluid with a density of 1.06 g cm\(^{-3}\) is flowing into the bioreactor at a flow rate of 125 L s\(^{-1}\) and this flow is evenly distributed into each of these fiber tubes. The viscosity of the fluid is 15 cP. Is the flow in each tube laminar or turbulent? Estimate the pressure drop in mmHg across the bioreactor.

4.77 The following data was obtained for the flow of a viscous fluid in a capillary tube: L = 50 cm, \mu = 5 \text{ cP}, Q = 0.3 \text{ cm}^3 \text{ s}^{-1}, and \Delta P = 0.375 \text{ mmHg}. Estimate the diameter (cm) of this capillary tube from these data.

4.78 A capillary viscometer was used to measure the viscosity of a fluid at room temperature. At a pressure drop per unit length of 0.28 mmHg cm\(^{-1}\), i.e., \Delta P/L, the measured flow rate of the fluid through the capillary viscometer was found to be 0.0125 cm\(^3\) s\(^{-1}\). When the value of \Delta P/L was increased to 0.79 mmHg cm\(^{-1}\), the flow rate of the fluid was found to be 0.047 cm\(^3\) s\(^{-1}\). Under these conditions, is the fluid Newtonian or non-Newtonian? Be sure to justify your answer.

4.79 A fire suppression system (FSS) for a tall building pumps water from a river and delivers the water through a long pipe to where the water flows on top of the water already in a storage tank at the top of the building. When the pump is running, the water flows into the storage tank at 30 L s\(^{-1}\). The water exits the pipe 30 m above the surface of the river. The pipe has an internal
diameter of 10 cm. What pumping power (HP) is needed to deliver the water? The viscosity of water is 1 cP and the density of the water is 1 g cm\(^{-3}\). Assume \(h_{\text{friction}} = 100.32\ m^2\ s^{-2}\).

**4.80** An absorbent material is being designed to clean up a toxic liquid material. The surface tension of the toxic material is 0.028 N m\(^{-1}\). The absorbent material has within its structure tiny cylindrical openings that have a diameter of 0.5 mm. The cylindrical openings cover 45% of the absorbent material’s surface. The contact angle (\(\theta\)) of the toxic substance and the absorbent material is such that the \(\cos \theta = 0.9105\). From this information estimate the initial mass flux (grams per second per cm\(^2\) of absorbent material) of the toxic material into the absorbent material. The density of the toxic substance is 1.04 g cm\(^{-3}\).

**4.81** A polymeric material is being extruded as a very thin ribbon of thickness 3 mm and width 25 mm. As part of the extrusion process, the material is to be treated with an antistatic agent as it leaves the extruder. This antistatic agent wicks into the porous space of the polymeric material. The radius of the pores in the polymeric material is 5 \(\mu\)m. It is desired that the antistatic agent penetrate 150 \(\mu\)m into the material. Estimate how long will it take for the antistatic agent to enter into the pores of the polymeric material. The antistatic agent has a viscosity of 38 cP and a surface tension of 30 milliN m\(^{-1}\). The cosine of the contact angle, i.e., \(\cos \theta\), is equal to 0.84.

**4.82** An adhesive material was studied in a viscometer. A linear regression equation based on the Casson equation was found to describe the data as: \(\tau^{1/2} = 0.238 \dot{\gamma}^{1/2} + 7.7\), where \(\tau\) is the shear stress in Pa and \(\dot{\gamma}\) is the shear rate in s\(^{-1}\). An adhesive applicator system is being designed that will dispense this adhesive material through a polymeric tube with an inside diameter of 2.3 mm and a total length of 100 cm. If the pressure drop over the length of this tube is 2000 mmHg, estimate the flow rate of the adhesive material through the tube in cm\(^3\) min\(^{-1}\).

**4.83** A drug injection system consists of a large chamber (length to diameter ratio = 10) that contains the drug solution and a small catheter tube that goes to the patient. This chamber contains a moving piston that separates the drug containing region from the region of the chamber where air is introduced to pneumatically drive the piston to induce flow of the drug solution. The air is maintained at a constant pressure during the drug injection process. The drug solution is injected into the patient at a flow rate of 10 cm\(^3\) s\(^{-1}\) through a catheter tube that has an inner diameter of 1 mm and a length of 15 cm. Also, the drug solution has a viscosity of 0.85 cP and a density of 1.02 g cm\(^{-3}\). The drug is injected into the abdominal cavity of the patient where the local pressure is 0 mmHg gauge. Assume the drug chamber is horizontal and that you can neglect any frictional force developed between the piston and the walls of the chamber containing the drug solution. Also, any pressure losses due to fluid motion within the drug solution chamber are negligible in comparison to the frictional losses within the tubing that leads from the drug solution chamber and into the patient. What air pressure (mmHg) is required to achieve this flow rate of the drug solution?

**4.84** Nutrient media is being transferred through a horizontal 2 in. diameter pipe from the feedstock building to the fermentation building at a flow rate of 60 gal min\(^{-1}\). Pressure gauges are installed in the pipe in each building to monitor the transfer operation. If the length of the transfer pipe between these two pressure gauges is 75 ft, what would be the expected pressure drop in mmHg? At the planned operating conditions the nutrient media has a viscosity of 0.85 cP and a density of 1 g cm\(^{-3}\).

**4.85** Experiments on a viscous biological fluid were done in a capillary viscometer having an internal diameter of 1.75 mm. The data for eight experiments when plotted as the volumetric flow rate, Q, versus the pressure drop over the length of the capillary tube, i.e., \(\Delta P/L\), showed a

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linear relationship between $Q$ and $\Delta P/L$. A regression line through these data with a zero intercept gave a slope of 83.5 mL cm min$^{-1}$ mmHg$^{-1}$ and the maximum flow rate for these experiments was $Q = 50$ mL min$^{-1}$. Based on this information, estimate the viscosity ($\mu$) of the solution in cP. Assume the density of this solution is about 1 g cm$^{-3}$.

4.86 The Carreau viscosity model has been shown to describe the apparent viscosity of blood (Lee et al., 2014). According to this model, at low shear rates the fluid is Newtonian and at high shear rates the fluid follows the power law:

$$\mu_{\text{apparent}} = \mu_\infty + (\mu_0 - \mu_\infty) \left[1 + (\lambda \dot{\gamma})^2\right]^\frac{n-1}{2}$$

In this equation, $\mu_{\text{apparent}}$ is the non-Newtonian apparent viscosity of the fluid, $n$ is the power law index, and $\lambda$ is the relaxation time. $\mu_0$ and $\mu_\infty$ are, respectively, the fluid viscosity at zero shear rate and at infinite shear rate. Using the data shown in Figure 4.5, find the best values of the parameters in the Carreau viscosity model.

4.87 Blood flows through a 75 $\mu$m diameter glass tube that is 10 cm in length. This particular blood has a core viscosity of 3.3 cP and a marginal zone layer thickness (or plasma layer) of 2.4 $\mu$m, i.e., $\delta$. Estimate the volumetric flow rate of the blood in cm$^3$ h$^{-1}$ if the pressure difference over the length of the tube is 8000 Pa. The plasma viscosity of this blood is 1.2 cP.

4.88 Using the Carreau viscosity model described in Problem 88, if the shear rate on a Carreau fluid is 3 s$^{-1}$, what is the shear stress on the fluid (Pa)? Use the following values for these parameters given by Lee et al. (2014):

- $\mu_0 = 0.056$ Pa s
- $\mu_\infty = 0.0035$ Pa s
- $\lambda = 3.313$ s
- $n = 0.3568$

4.89 An FSS at a remote resort in Jasper, Alberta, is being designed to take water from a river and send it through a hose where it exits through a nozzle at a turbulent velocity of 30 m s$^{-1}$. The pump must be able to deliver 2000 L min$^{-1}$ of water. The inside diameter of the hose is 0.10 m and the equivalent length of the hose, which includes the losses due to entrance and exit effects, valves, and fittings, is 90 m. Also, the FSS must be able to lift the water 30 m from the river surface to where it exits the nozzle. How much power must the pump in the FSS have to meet these conditions? The viscosity of water is 1 cP and its density is 1 g cm$^{-3}$.

4.90 Derive Equation A in Example 4.1.

4.91 A capillary viscometer is being used to measure the viscosity of a liquid at 20 C. The capillary tube is 0.01 cm in diameter and has a length of 100 cm. When the pressure drop over the length of this capillary tube was 2 atm, the flow rate of the liquid was found to be 1.0 cm$^3$ h$^{-1}$. What is the viscosity of the liquid in cP? The density of the liquid is 1.025 g cm$^{-3}$.

4.92 Two tanks are connected by 300 m of 7.5 cm diameter steel pipe. One of the tanks is open to the atmosphere (tank 2) and the other tank (tank 1) is maintained at an internal pressure of $P_1$. The diameter of each tank is quite large so that the velocities of the liquid surface in each tank is negligible. The fluid in the tanks is an oil with a viscosity of 100 cP and a density of 0.80 g cm$^{-3}$. What should the pressure be in the closed tank, i.e., $P_1$, relative to atmospheric pressure, i.e., $P_2$, so that the flow rate of the oil from tank 1 to tank 2 is 7 kg s$^{-1}$? Express the pressure in units of mmHg. The surface of the liquid contained in tank 1 lies 9 m below the surface of the liquid in tank 2.
Author Queries

[AQ1] Style for acronyms and their expansions is inconsistent in the manuscript. We have followed the House Style in this regard, i.e., defining an acronym at its first instance (of every chapter) and thereafter using the acronym alone in that chapter. Please check if this is okay.

[AQ2] Please provide captions for parts (a), (b) and (c) in Figures 4.1, 4.10.

[AQ3] Please check if cross reference for Equation 4.2a and b are correct.

[AQ4] Both “$U$” and “$\bar{U}$” have been used. Please check if okay.

[AQ5] Please provide details of Bayliss (1952) in the reference list.

[AQ6] Please check if bold font for numbers in Table 4.4 is appropriate.

[AQ7] Please check the sentence beginning with “In this equation, $H$ represents...” for clarity.

[AQ8] Please check if Owen et al. (2000) can be set as “source” to the following table.

[AQ9] Please provide article title for “Bush, Petros, and Barrett-Lennard (J. Biomech., 30, 967–969, 1997)” and confirm if this reference can be added to the book-end reference list.

[AQ10] Both “viscometry data” and “viscometer data” have been used in the text. Please check if one form should be made consistent.

[AQ11] Please check if “Problem 88” in the sentence “Using the Carreau viscosity model described in Problem 88...” is correct.